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Gleason's Theorem in W^*J-Algebras in Spaces with Indefinite Metric

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We characterize measures on hyperbolic logics associated to von Neumann algebras acting in a space with an indefinite metric. An analog to the Gleason theorem is proved.

1. INTRODUCTION

The problem of describing measures on logics is well known (see ref. 2 or ref. 3 Chapter XII) . A celebrated theorem of Gleason[6] serves as a basis for the quantum measure theory. The theorem asserts that every probability measure μ on the orthogonal projections Π on a Hilbert space $\mathcal{H}$ with dim $\mathcal{H} \geq 3$ is of the form $\mu(p) = \mathrm{tr}(Tp)$, where $T \geq 0$ is a uniquely determined trace-class operator. The problem of describing all probability measures on projections arose in noncommutative probability theory. For von Neumann algebras of type II, an analog to the Gleason theorem was proved in ref. 9. Three years later this result was reproduced by Yeadon[16] with a similar proof. The case of type III was examined in ref. 3 and ref. 10 independently. An analog to the Gleason theorem for charges (= real measures) was obtained in 11. There has been significant progress for the logic L of all (skew) projections in a von Neumann algebra. In refs. 15 and 12 we proved that for signed measure (= charge) ν on the Logic L of all (not necessarily orthogonal) projections in a semifinite von Neumann algebra containing no central summand of type I_2, the following generalization to the Gleason formula holds: $\nu(p) = \Re\, tr(Tp)$, $\forall p \in L$, where T is a trace-class operator.

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Theorem. Let P be the logic of all J-self-adjoint projections in a W^*J-algebra $\mathcal{A}$ acting in a space with an indefinite metric containing no central summand of type $I_{n,m}$ (n, $m \leq 2$). Then for every indefinite measure ν: $P \to R$ there is a J-self-adjoint trace-class operator T such that:

(i) If $\mathcal{A}$ is a W^*P-algebra, then

$$\nu(p) = tr(Tp) + \nu_0(p), \qquad \forall p \in P$$

for some semiconstant ν_0

(ii) If $\mathcal{A}$ is a W^*K-algebra, then

$$\nu(p) = tr(Tp), \qquad \forall p \in P \tag{1}$$

In the present paper, we characterize measures on hyperbolic logics associated to algebras of operators acting in a space with an indefinite metric. Note that the problem of the construction of a quantum field theory leads to indefinite metric spaces.(3) As is well known, for the logic Π (as well as for the logics L), no general approach to describing measures has been found that is suitable for all continuous algebras. Two different methods have been suggested for algebras of types II and III. For the logics P, we suggest a common approach.

We present the necessary definitions and notation. Let H be a Hilbert space with an inner product (., .). Let J be a linear or conjugate linear invertible bounded operator on H. Put $[x, y] = (Jx, y)$, $\forall x, y \in H$. Let $\mathcal{A}$ be an algebra of bounded operators in H with the unit I closed in the weak operator topology and closed with respect to the J-conjugation, i.e., if $a \in \mathcal{A}$, then $a^0 \in \mathcal{A}$, where a^0 is a bounded operator such that $[ax, y] = [x, a^0 y]$, $\forall x, y \in H$. Such an algebra is called a J-algebra. Denote by P [$= P(\mathcal{A})$] the set of all J-self-adjoint projections in $\mathcal{A}$, i.e., $P = \{p \in \mathcal{A}: p^2 = p, [px, z] = [x, pz], \forall x, z \in H\}$. With respect to the ordering $p \leq q \Leftrightarrow pq = qp = p$, to the orthocomplementation $p \to p^{\perp} \equiv I - p$, and to the orthogonal relation $p \perp q \Leftrightarrow pq = qp = 0$, the set P is a quantum logic.

In general, P is not a lattice or a σ-logic.

There have been many studies of J-self-adjoint operators if J is a self-adjoint (in the Hilbert space H) operator, $J \neq \pm I$, $J^2 = I$. Below, we will consider this case. There exist orthogonal projections Q^+ and Q^- such that $Q^+ + Q^- = I$, $J = Q^+ - Q^-$. Put $H^+ = Q^+ H$ and $H^- = Q^- H$. According to the terminology of ref. 1 [. . .] is an indefinite metric in H, J is a canonical symmetry, $H = H^+ [+] H^-$ is a canonical decomposition, and H is a Krein space (sometimes H is called a J-space).

Let $p \in B(H)$. It is easy to see that $[px, y] = [x, py]$, $\forall x, y \in H \Leftrightarrow p = Jp^*J$ ($= p^0$).

Let S be the unit sphere in the Hilbert space H. The set $\Gamma \equiv \{r \in H: [r, r]^2 = 1\}$ is an indefinite analog of the unit sphere. It is easy to see that every one-dimensional projection $p \in P$ can be represented in the form $p_f = [f, f][\cdot, f]f, f \in \Gamma$. Suppose that $H = R^3$ and $\mathcal{A} = B(H)$. Then Γ is the union of two hyperboloids, $\{(x, y, z): x^2 + y^2 - z^2 = 1\}$ and $\{(x, y, z): z^2 - x^2 - y^2 = 1\}$. Therefore, in this case the logic P could be called hyperbolic.

Let $\mathcal{A}_1$ be the set of all J-algebras in H. A J-algebra $\mathcal{A}$ is said to be a von Neumann J-algebra if $\mathcal{A}$ is, in addition, a von Neumann algebra. Denote by $\mathcal{A}_2$ the set of all von Neumann J-algebras. A von Neumann J-algebra $\mathcal{A}$ is called a W^*J-algebra if $J \in \mathcal{A}$ and the cenral covers of Q^+ and Q^- equal I. Let $\mathcal{A}_3$ be the set of all W^*J-algebras in H. Obviously $\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1$. We say that a W^*J-algebra $\mathcal{A}$ has the type I (II, III) if the W^*-algebra $\mathcal{A}$ has the type I (II. III). We say that a W^*J-algebra $\mathcal{A}$ is a W^*P-algebra if at least one of the projections Q^+ and Q^- is finite (with respect to $\mathcal{A}$). A W^*J-algebra $\mathcal{A}$ is said to be a W^*K-algebra if the W^*-algebras $Q^+\mathcal{A}Q^+$ and $Q^-\mathcal{A}Q^-$ contain no nonzero finite direct summand. For every W^*J-algebra $\mathcal{A}$ there exist three central projections E_+, E_-, and E such that $E_+ + E_- + E = I$, Q^+E_+ is a finite projection with respect to the W^*-algebra $\mathcal{A}E_+$, Q^-E_- is finite with respect to $\mathcal{A}E_-$, and $\mathcal{A}E$ is a W^*K-algebra. Note that every W^*P-algebra inherits properties of $B(H)$ in a Pontryagin space and every W^*K-algebra inherits properties of $B(H)$ in a Krein space with $\min\{\dim H^+, \dim H^-\} = \infty$. We say that a W^*J-algebra $\mathcal{A}$ is of type $I_{n,m}$ if $Q^+\mathcal{A}Q^+$ and $Q^-\mathcal{A}Q^-$ (in $Q^+ H$ and in $Q^- H$) is of type n and m, respectively.

Elementary properties. Let $\mathcal{B}$ be a von Neumann algebra acting in a Hilbert space $\mathcal{H}$. Let $\mathcal{L}$ and $\mathcal{B}^p$ be the set of all projections and the set of all orthogonal projections in $\mathcal{B}$. There exist P which are isomorphic to L or $\mathcal{B}^p$.[(14)]

A specific character of J-spaces becomes fully transparent when considering the logic P for a W^*J-algebra $\mathcal{A}$. We shall consider this case. Denote by Π the set of all orthogonal projections in $\mathcal{A}$. Now, let P^+ (P^-) be the set of all projections $p \in P$ for which pH is *positive*, i.e., $\forall x \in pH$, $x \neq 0$, $[x, x] > 0$ (*negative*, i.e., $\forall x \in pH$, $x \neq 0$, $[x, x] < 0$). Note that $p \in P^+$ ($p \in P^-$) $\Leftrightarrow p^* \in P^+$ ($p^* \in P^-$) $\Leftrightarrow Jp \geq 0$ ($Jp \leq 0$). Every $e \in P$ is representable (not uniquely) as $e = e_+ + e_-$, where $e_+ \in P^+$, $e_- \in P^-$.

A sum $e = \Sigma e_i$ for $e_i \in P$, $e_i \perp e_j$ $(i \neq j)$ is said to be a *decomposition* of e (the sum shuld be understood in the strong sense). A mapping $\mu : P \to R$ is called a *measure* if $\mu(e) = \Sigma\mu(e_i)$ for every decomposition $e = \Sigma e_i$.

Here, the convergence of an uncountable family of summands means that there exists only a countable set of nonzero terms in the family and the usual series with these summands converges absolutely.

A measure is said to be *indefinite* if $\mu/P^+ \geq 0$ and $\mu/P^- \leq 0$; *linear* if (1) holds; and a *semiconstant* (=*semitrace*) if $\mu(p) = c\tau(Ep_+)$, $\forall p \in P$, or $\mu(p) = c\tau(Ep_-)$, $\forall p \in P$. where τ is a faithful normal semifinite trace on

$\mathcal{A}$ and E is an operator affiliated to the center of $\mathcal{A}$. We call a measure μ *Hermitian* if $\mu(p) = \mu(p^*)$, $\forall p \in P$, and *skew-Hermitian* if $\mu(p) = -\mu(p^*)$, $\forall p \in P$. Every measure μ can be represented as the sum of the Hermitian component $\mu_h(p) \equiv 1/2[\mu(p) + \mu(p^*)]$ and the skew-Hermitian one $\mu_s(p) \equiv 1/2[\mu(p) - \mu(p^*)]$. Clearly, if μ is an indefinite measure, then its Hermitian component μ_h is an indefinite measure also.

Remark 1. An indefinite measure is an analog for a probability measure on the logic P. In ref. 13 we proved that for any indefinite measure μ in a Krein space H, dim $H \geq 3$ and for the W^*J-algebra $B(H)$ the main theorem is true.

We need the following concept of a variation of a measure μ; $\|\mu\|(p) \equiv \sup\{\Sigma|\mu(p_i)|\}$ taken over all possible decompositions $p = \Sigma\, p_i$. Also, put

$$M_\alpha^\mu \equiv \sup\{|\mu(p)|: p \in P^+, \|p\| \leq 2\alpha - 1, \alpha > 1\} \tag{1}$$

2. THE STRUCTURE OF THE PROJECTIONS IN *P*

For any operator $x \in \mathcal{A}$ denote by F_x the orthogonal projection onto $\overline{xH}$. Let $e, f \in \Pi$. We write $e \sim f$ if there exists a partial isometry $v \in \mathcal{A}$ with the initial projection e and the final projection $F_v = f$. We write $e \preccurlyeq f$ if $F_v \leq f$. Without loss of generality it can be assumed that $Q^+ \preccurlyeq Q^-$(ref. 5, Theorem 1, p. 218). Denote by V the set of all partial isometries $v \in \mathcal{A}$ with the initial projection not exceeding Q^+ and with the final projection $F_v \leq Q^-$.

Proposition 2. For every $p \in P^+$ we have

$$p = x + v(x^2 - x)^{1/2} - (x^2 - x)^{1/2}v^* - v(x - F_x)v^* \tag{2}$$

where $x \equiv Q^+pQ^+$ $(\geq F_x)$ and v is a partial isometry in the polar decomposition $Q^-pQ^+ = v|Q^-pQ^+|$. Conversely, let $x \in \mathcal{A}$ be an arbitrary operator such that $x \geq F_x$ and $F_x \leq Q^+$, and let $v \in V$ be an arbitrary partial isometry with the initial projection F_x. Then (2) defines a projection in P^+.

Proof. See ref. 14.

By the symmetry, every $q \in P^-$ has the following representation:

$$q = z + w(z^2 - z)^{1/2} - (z^2 - z)^{1/2}w - w(z - F_z)w^* \tag{3}$$

where $z \equiv Q^-qQ^-$ $(\geq F_z)$ and w is the partial isometry in the polar decomposition $Q^+qQ^- = w|Q^+qQ^-|$ for Q^+qQ^-.

For any projection $p \in P$ we denote by e_p the orthogonal projection onto $\overline{Q^+pH}$. We say that a projection $p \in P$ is simple if $e_pF_pe_p = \alpha e_p$ and $F_pe_pF_p = \alpha F_p$, $\alpha \in (0.1)$. Note that every simple projection is either positive or negative. The following corollary is a straightforward consequence from Proposition 2.

Corollary 3. Suppose that $p \in P^+$ is represented by (2). Then we obtain:
(i) $\|p\| = \|2x - I\| = 2\|x\| - 1$
(ii) $F_p = x\,(2x - I)^{-1} + v(x^2 - x)^{1/2}(2x - I)^{-1} + (2x - I)^{-1}(x^2 - x)^{1/2}v^* + v(x - F_x)(2x - I)^{-1}v^*$
(iii) The projection p is simple if and only if $x = \alpha F_x$, $\alpha > 1$.
We mention one more property:
(iv) If an orthogonal projection $e \in \Pi$ is such that $eQ^+e = \beta e$, $\beta \in (1/2, 1)$, then there is a simple projection $p \in P^+$ such that $e = F_p$ and $\|p\| = (2\beta - 1)^{-1}$.

Remark 4. Every $p \in P^+ \cup P_-$ can be approximated in the norm with a sum of mutually orthogonal simple projections.

In the sequel, the projection p of the form (2) will be denoted by $p(x, v)$ and the projection q of the form (3) by $q(w(z - F_z)w^*, w^*)$.

Lemma 5. Let $p \in P^+$ and $e_p \preccurlyeq Q^- \wedge (F_p \vee e_p)^\perp$. Then there exists a simple projection $g \in P^+$ such that:
1. $e_p = e_g$, $\|e_g - g\| \leq \|e_p - p\|$.
2. In the Hilbert space H with the norm $\|\cdot\|_1$ generated by a new canonical symmetry $J_1 = Q_1^+ - Q_1^- \in \mathcal{A}$ such that $p \leq Q_1^+$, the projection g is simple and $Q_1^+\, gH = pH$, $\|g - p\|_1 \leq \|e_p - p\|$.

Proof. One can suppose that $Q^+H \cap pH = 0$. Let $p = p(x, v)$ and $\alpha \equiv 1/2(\|p\| + 1)$ $(=\|x\|)$. By Corollary 3 and Proposition 2, $e_p < x \leq \alpha e_p$. Put

$$y_0 \equiv (\alpha - 1)^{-1}(x - e_p)\,\{\alpha^{1/2}I + [\alpha e_p - (x - e_p)]^{1/2}\}^{-2}$$

Thus $0 < y_0 \leq e_p$. By the assumption, there exists a partial isometry $w \in \mathcal{A}$ with the initial projection vv^* and the final one $F_w \leq Q^- \wedge (F_p \vee e_p)^\perp$. Let

$$z \equiv vy_0^{1/2}\,v^* + w(F_v - vy_0^{1/2}v^*)^{1/2} \quad [= vy_0^{1/2}v^* + wv(e_p - y_0)^{1/2}v^*]$$

It can be easily shown that z is a partial isometry with the initial projection $vv^* = F_v$. By the construction, $g \equiv p(\alpha e_p, zv)$ is a simple projection, $e_g = e_p$, and $\|e_g - g\| = \|e_p - p\|$. The operator $y_0^{1/2}$ is a solution of the equation

$$\alpha(x - e_p)^{1/2} = 2[\alpha(\alpha - 1)]^{1/2}\,y^{1/2} - (\alpha - 1)(x - e_p)^{1/2}y$$

Making use of this, we can verify that

$$pgp = p(x, v)p(\alpha e_p, zv)p(x, v) = \alpha p(x, v)$$

Let $J_1 = Q_1^+ - Q_1^- \in \mathcal{A}$ be a new canonical symmetry, where $p \leq Q_1^+$, with respect to a new canonical decomposition $H = H_1^+\,[+]\,H_1^-$ (see Definitions,

§3, Chapter I, ref. 1). The new Hilbert product $(x, y)_1 \equiv [J_1 x, y]$ is equivalent to the product (., .) in H (see Theorem 7.19, §7, Chapter I, ref. 1). Hence the projection $p(\alpha e_p, zv)$ is simple in the Hilbert space H with $(., .)_1$. By the construction, $\|p - g\|_1 = \|e_p - p\|$.

Let $e \in \Pi$ be a projection such that 1/2 is a regular point for eQ^+e. (In this case, eH is a Krein space[1] with respect to the metric [., .], and there exists a projection $p \in P$ such that $e = F_p$.) Let $eQ^+e = \int \lambda \, de_\lambda$ be the spectral decomposition of eQ^+e. Put $p_t = (\int_{-0}^{t} (2\lambda - 1)^{-1} \, d(ee_\lambda))J$, $\forall t > 0$ [here $(1/0) \cdot 0 \equiv 0$]. By the definition, $Jp_t^* J = p_t$, $\forall t$. We have

$$(e_s - e_t)J(e_\lambda - e_\beta) = (e_s - e_t)(2Q^+ - I)(e_\lambda - e_\beta)$$
$$= e(2Q^+ - I)(e_s - e_t)(e_\lambda - e_\beta) = eJ(e_s - e_t)(e_\lambda - e_\beta)$$

$\forall s, t, \lambda, \beta \in (0, +\infty)$. This means that $p_t^2 = p_t$, $\forall t$. In addition, $p_{t_2} - p_{t_1} \in P^+$, $\forall t_1, t_2 \in (1/2, 1]$, $t_1 < t_2$, and $p_{t_2} - p_{t_1} \in P^-$, $t_1, t_2 \in [0, 1/2]$, $t_1 < t_2$. Thus

$$p_t \in P, \quad \forall t, \qquad \text{and} \qquad e = \left(\int_{0-}^{1+} (2t - 1) \, dp_t \right) J$$

3. SOME PROPERTIES OF A LINEAR SKEW-HERMITIAN MEASURE

Let $\mathcal{A}_*$ be the set of all norm-continuous linear functionals on $\mathcal{A}$. Let $\phi \in \mathcal{A}_*$ be such that $\phi(.)$: $P \to R$ is an indefinite measure and $(\phi.J)(b) \equiv \phi(bJ)$, $\forall b \in \mathcal{A}$. By (4), $(\phi.J)(e) = \phi((\int_{0-}^{1+} (2t - 1)dp_t)JJ) \geq 0$, for every orthogonal projection $e \in \Pi$ for which 1/2 is a regular point of eP^+e. Hence $(\phi.J)(\cdot)$ is a nonnegative linear functional. Let $\psi \in \mathcal{A}_*$. Then the functionals $\psi^0(x) \equiv \psi(x^0)$ and $\psi^*(x) \equiv \psi(x^*)$, $\forall x \in \mathcal{A}$, belong to $\mathcal{A}_*$. A functional ψ is said to be J-self-adjoint (J-skew-adjoint) if $\psi = \psi^0$ ($\psi = -\psi^0$).

Let μ: $P \to R$ be a linear measure and let $f \in \mathcal{A}_*$ be such that $\mu(p) = f(p)$, $\forall p \in P$. Then the functional $\psi \equiv 1/2(f + f^0)$ is J-self-adjoint and $\mu(p) = \psi(p)$, $\forall p \in P$. It is clear that $x^{*0} = x^{0*}$, $\forall x \in \mathcal{A}$. Hence (1) $f^{*0} = f^{0*}$, $\forall f \in \mathcal{A}_*$, (2) if $f = f^0$, then $f^* = f^{*0}$, and (3) if $f = f^*$, then $f^0 = f^{0*}$. Thus (i) if μ is a Hermitian measure, then there is a self-adjoint and J-self-adjoint functional f_h $[= 1/4(f + f^* + f^0 + f^{0*})]$ such that $\mu(p) = f_h(p)$, $\forall p \in P$, (ii) if μ is a skew-Hermitian measure, then there is a skew-adjoint and J-adjoint functional f_s $[= 1/4(f - f^* + f^0 - f^{0*})]$ such that $\mu(p) = f_s(p)$, $\forall p \in P$. For every self-adjoint functional $f \in \mathcal{A}_*$ there exist two positive normal functionals $f_+, f_- \in \mathcal{A}$ with mutually orthogonal covers e_+ and e_- such that $f = f_+ - f_-$ (ref. 5, Theorem 6, p. 64).

Proposition 6. Let $f \in \mathcal{A}_*$ and $f = f^* = f^0$. Then $e_+ J = Je_+$, $e_- J = Je_-$, and, furthermore, $f = f(Q^+.Q^+) + f(Q^-.Q^-)$.

Proof. It is clear that

$$\begin{aligned} 0 \le f_+(e_+) &= f(e_+) = f^0(e_+) = f(Je_+J) \\ &= f_+(e_+(Je_+J)e_+) - f_-(e_-(Je_+J)e_-) \le f_+(e_+) \end{aligned}$$

Thus $f_+(e_+(Je_+J)e_+) = f_+(e_+)$. Hence $e_+(I - Je_+J)e_+ = 0$, i.e., $e_+ \le Je_+J$. Then $Je_+J \le J(Je_+J)J = e_+$. Finally, $e_+ = Je_+J$. Similarly, $e_- = Je_-J$. In addition.

$$\begin{aligned} \overline{f(Q^- \ a^* Q^+)} &= f^*(Q^+ a Q^-) = f(Q^+ \ aQ^-) = f^0(Q^+ \ aQ^-) \\ &= \overline{f(JQ^- \ a^* Q^+ J)} = -\overline{f(Q^- a^* Q^+)} \end{aligned}$$

$\forall a \in \mathcal{A}$. Hence $f(Q^+ \ aQ^-) = 0$ and, similarly $f(Q^- \ aQ^+) = 0$, $\forall a \in \mathcal{A}$. This means $f = f(Q^+. \ Q^+) + f(Q^-. \ Q^-)$.

For any $f \in \mathcal{A}_*$ and $u \in \mathcal{A}$ denote by $(u.\ f)$ $[(f.\ u)]$ the functional $f(u.)$ $[f(.u)$, respectively].

Proposition 7. Let $f \in \mathcal{A}_*$ and $f = f^0 = -f^*$. Let $(Q^-.f.Q^+) = (|Q^-.f.Q^+|.u)$ be the polar decomposition of the functional $(Q^-.f.Q^+.)$ (Theorem 4, p. 61 ref. 5). Then the following formula is true:

$$f = f(Q^-.Q^+) + f(Q^+.Q^-) = (|Q^-.f.Q^+|.u) - (u^*.|Q^-.fQ^+|)$$

Proof. For every orthogonal projection $e \in \mathcal{A}$ with $e \le Q^+$ or $e \le Q^-$ the following is true:

$$\overline{f(e)} = \overline{f(JeJ)} = \overline{f(e^0)} = f^0(e) = -f^*(e) = -\overline{f(e)}$$

Hence

$$f(.) = f(Q^-.Q^+) + f(Q^+.Q^-) \equiv (Q^-.f.Q^+)(.) + (Q^+.f.Q^-)(.)$$

Thus

$$\begin{aligned} \overline{f(a)} &= \overline{f^0(a)} = f(Ja^*J) \\ &= f(Q^-(Ja^*J)Q^+) + f(Q^+(Ja^*J)Q^-) = -f(Q^- a^* Q^+) - f(Q^+ a^* Q^-) \end{aligned}$$

$\forall a \in \mathcal{A}$. But

$$\begin{aligned} (Q^+.f.Q^-)(a) &= f(Q^+ a Q^-) = -f^*(Q^+ a Q^-) \\ &= -\overline{f(Q^- a^* Q^+)} = -(Q^-.f.Q^+)^*(a) \end{aligned}$$

i.e., $-(Q^-.f.Q^+)^* = (Q^+.f.Q^-)$. Let $(Q^-.f.Q^+) = (|Q^-.f.Q^+|.u)$ be the polar decomposition of the functional $(Q^-.f.Q^+)$. Then

$$f = (Q^- .f.Q^+) + (Q^+ .f.Q^-) = (Q^- .f.Q^+) - (Q^- .f.Q^+)^*$$
$$= (|Q^- .f.Q^+|.u) - (u^* .|Q^- .f.Q^+|)$$

Let $\phi \in \mathcal{A}_*$ be a nonnegative normal linear functional with the support $e \leq Q^+$. Let $u \in V$ be a partial isometry with the initial projection e and, the final one $F_v \leq Q^-$. We define a linear functional by the formula

$$\overline{\mu}(a \equiv ((u^*.\phi) - (\phi.u))(a) = \phi(u^*a - au), \qquad \forall a \in \mathcal{A} \tag{5}$$

It is clear that $\overline{\mu}(p) \in R$, $\forall p \in P$. Hence $\overline{\mu}$ is a linear measure. Also, $\overline{\mu} = \overline{\mu}^0 = -\overline{\mu}^*$. Hence by Proposition 7, (5) gives the general form of a linear skew-Hermitian measure. It is clear that

$$\overline{\mu}(p) = \phi((x^2 - x)^{1/2} v^* u + u^* v (x^2 - x)^{1/2}), \qquad \forall p = p(x, v)$$

We define the functional $\phi_v(\cdot) \equiv 1/2\phi(u^*v. + .v^*u)$, where $v \in V$, and the skew-Hermitian measure $\overline{\mu}_v(p \equiv \phi_v(v^*p - pv)$, $\forall p \in P$. It is easy to see that

$$\overline{\mu}_v(p) = 1/2\phi(u^*p - u^*vpv + v^*pv^*u - pu)$$
$$= 1/2\overline{\mu}(p) + 1/2\phi(v^*pv^*u - u^*vpv)$$

$\forall p \in P$. Obviously, $\overline{\mu}(p) = \overline{\mu}_v(p)$, $\forall p = p(x, v)$.

Now, we adduce some properties of a linear skew-Hermitian measure that are crucial in the proof of the theorem.

Proposition 8. Let a skew-Hermitian measure $\overline{\mu}$ be defined by (5). Then the following properties hold:

(i) $M_\alpha^{\overline{\mu}} = \sup\{\|\overline{\mu}\|(p): \quad \|p\| \leq 2\alpha - 1\} = 2(\alpha^2 - \alpha)^{1/2}\phi(Q^+) = \mu(p(\alpha Q^+, u))$.

(ii) $\overline{\mu}(p(x, iu)) = 0$, $\forall x$; if $F_u \perp F_v$, then $\overline{\mu}(p(x, v)) = 0$.

(iii) If, for a given $\epsilon > 0$, $v \in V$ is chosen so that $M_\alpha^{\overline{\mu}} - \overline{\mu}(p(\alpha Q^+, v)) < \epsilon$, then

$$\left|\overline{\mu}(p) - \overline{\mu}_v(p)\right| \leq 2\|p\|(\|\phi\|\epsilon)^{1/2}(\alpha^2 - \alpha)^{-1/4}, \qquad p \in P$$

Proof. Properties (i) and (ii) follow directly from the definitions of $\overline{\mu}$ and $p(x, v)$. Let $v \in V$ be an partial isometry satisfying (iii). Then

$$\epsilon > M_\alpha^{\overline{\mu}} - \overline{\mu}_v\,(p(\alpha Q^+, v)) = (\alpha^2 - \alpha)^{1/2}\phi\,(2Q^+ - v^*u - u^*v) \geq 0$$

Also, $Q^- p^* Q^+ = Q^- JpJQ^+ = -Q^- pQ^+$, $\forall p \in P$. Hence

$$
\begin{aligned}
\phi(u^*p - v^*pv^*u) &= \phi(u^*Q^-pQ^+ - v^*Q^-pQ^+v^*u) \\
&= \phi(v^*Q^-p^*Q^+v^*u - u^*Q^-p^*Q^+) \\
&= \overline{\phi(u^*vQ^+pQ^-v - Q^+pQ^-u)} \\
&= \overline{\phi(u^*vpv - pu)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\left|\overline{\mu}(p) - \overline{\mu}_v(p)\right| \\
&\quad = 1/2\left|\overline{\mu}(p) - \phi(v^*pv^*u - u^*vpv)\right| \\
&\quad = 1/2\left|\phi(u^*p - pu) - \phi(v^*pv^*u - u^*vpv)\right| \\
&\quad = 1/2\left|\phi(u^*p - v^*pv^*u) + \phi(u^*vpv - pu)\right| \\
&\quad \leq \left|\phi(u^*p - v^*pv^*u)\right| \\
&\quad = \left|\phi(u^*p - u^*pv^*u + u^*pv^*u - v^*pv^*u)\right| \\
&\quad \leq \left|\phi(u^*p(Q^+ - v^*u))\right| + \left|\phi((u^* - v^*)pv^*u)\right| \\
&\qquad\qquad \text{(by the Schwarz inequality)} \\
&\quad \leq \phi(u^*pp^*u)^{1/2}\phi((Q^+ - u^*v)(Q^+ - v^*u))^{1/2} \\
&\qquad + \phi(u^*vp^*pv^*u)^{1/2}\phi((u^* - v^*)(u - v))^{1/2} \\
&\quad \leq \phi(Q^+)^{1/2}\|p\|(\phi(Q^+ - u^*v - v^*u + Q^+)^{1/2} \\
&\qquad + \phi(Q^+ - v^*u - u^*v + Q^+)^{1/2}) \\
&\quad = 2\|p\|\phi(Q^+)^{1/2}(\phi(2Q^+ - u^*v - v^*u))^{1/2} \qquad \text{[by (i)]} \\
&\quad = 2\|p\|\phi(Q^+)^{1/2}((M_\alpha^{\overline{\mu}} - \overline{\mu}_v(p(\alpha Q^+, v)))(\alpha^2 - \alpha)^{-1/2})^{1/2} \\
&\quad \leq 2\|p\|(\|\phi\|)\epsilon)^{1/2}(\alpha^2 - \alpha)^{-1/4}
\end{aligned}
$$

The proof is complete.

4. REDUCING THE DESCRIPTION PROBLEM TO A SKEW-HERMITIAN MEASURE

Proposition 9. Let $\nu(p) = \tau(Ep_+)$, $p \in P$, be a semiconstant measure. Then $\nu(e_p) = \tau(EF_p) = \nu(p)$, $\forall p \in P^+$.

Proof. Let $p \in P^+$. By the definition of a semiconstant measure, the operator E is affiliated to the center of $\mathscr{A}$. By the definition of e_p, there exists a partial isometry $v \in \mathscr{A}$ with the initial projection F_p and the final one e_p. Hence

$$\nu(e_p) = \tau(Ee_p) = \tau(Evv^*) = \tau(Ev^*v)$$
$$= \tau(EF_p) = \tau(E(F_p + F_p p F_p^{\perp}) = \tau(Ep) = \nu(p)$$

Recall that without loss of generality we may suppose $Q^+ \preccurlyeq Q^-$.

Lemma 10. Let an indefinite measure ν: $P \to R$ be representable as a sum of two measure: $\nu = \nu_1 + \nu_0$, where ν_0 is a semiconstant measure and the restriction of ν_1 to every W^*J-factor $\mathcal{B} \subset \mathcal{A}$ of type I_2 is linear. Then

$$\left|\nu(e_p) - \nu(p)\right| \leq 6\,\|e_p - p\|(1 - 2)\|e_p - p\|)^{-1}(\nu(Q^+) - \nu(Q^-) + 5\|\nu_0\|(I))$$

$\forall p \in P^+$, $\|e_p - p\| < 1/2$.

Proof. (i) Let $\mu(.) \equiv tr(B.)$, where $BJ \geq 0$ be an indefinite measure. Then

$$\left|\mu(e) - \mu(p)\right| = \left|tr(BJJ(e - p))\right| \leq \|e - p\| tr(BJ)$$
$$= \|e - p\|(tr(BQ^+) - tr(BQ^-)) = \|e - p\|(\mu(Q^+) - \mu(Q^-)), \qquad \forall e, p \in P$$

Let $p \in P^+$ and $\|e_p - p\| < 1/2$.

1. We first assume that $e_p \preccurlyeq Q^- \wedge (F_p \vee e_p)$. Let $g \in P^+$ be as in Lemma 5. Denote by r_e the orthogonal projection onto $Q^- eH$, $\forall e \in P$, and by $\mathcal{A}(e_p, g)$ and $\mathcal{A}(p, g)$ the smallest J-self-adjoint algebras generated by e_p, g and p, g, respectively. By the construction, $\mathcal{A}(e_p, g)$ and $\mathcal{A}(p, g)$ are W^*J-factors of type I_2.

(a) Let $\nu_0 \equiv 0$. By the linearity of ν on $\mathcal{A}(e_p, g)$ and (i), we have

$$\left|\nu(r_p \vee r_g) - \nu(e_p + r_p \vee r_g - g)\right|$$
$$= \left|\nu(r_p \vee r_g - r_g) + \nu(r_g) - \nu(r_p \vee r_g - r_g) - \nu(e_p + r_g - g)\right|$$
$$\leq \|r_g - (e_p + r_g - g)\|(\nu(e_p) - \nu(r_g))$$
$$\leq \|e_p - p\|(\nu(e_p) - \nu(r_g)) \leq \|e_p - p\|(\nu(e_p) - \nu(r_g \vee r_p))$$

By (i).

$$\left|\nu(e_p) - \nu(g)\right| \leq \|e_p - p\|(\nu(e_p) - \nu(r_g)) \leq \|e_p - p\|(\nu(e_p) - \nu(r_g \vee r_p))$$

It is clear that $e_p + r_g \vee r_p - g \in P^-$. Thus

$$0 \leq \nu(g) - \nu(e_p + r_p \vee r_g - g)$$
$$\leq \nu(g) - \nu(e_p) + \nu(e_p) - \nu(r_p \vee r_g)$$
$$+ \nu(r_p \vee r_g) - \nu(e_p + r_p \vee r_g - g)$$
$$\leq (1 + 2\|e_p - p\|)(\nu(e_p) - \nu(r_p \vee r_g)) \tag{6}$$

Analogously, from the linearity of ν on $\mathcal{A}(p, g)$ we obtain

$$\max\{|\nu(p) - \nu(g)|, |\nu(e_p + r_p \vee r_g - g) - \nu(e_p + r_p \vee r_g - p)|\}$$
$$\leq \|e_p - p\|)(\nu(p) - \nu(e_p + r_p \vee r_g - p))$$

Thus

$$\begin{aligned}(1 - 2\|e_p - p\|)[\nu(p) - \nu(e_p + r_p \vee r_g - p)] \\ \leq [\nu(p) - \nu(e_p + r_p \vee r_g - p)] - [\nu(p) - \nu(p)] \\ - [\nu(e_p + r_p \vee r_g - g) - \nu(e_p + r_p \vee r_g - p)] \\ = \nu(g) - \nu(e_p + r_p \vee r_g - g)\end{aligned} \tag{7}$$

Hence by (6) and (7), we have

$$\nu(p) - \nu(e_p + r_p \vee r_g - p)$$
$$\leq (1 + 2\|e_p - p\|)(1 - 2\|e_p - p\|)^{-1}(\nu(e_p) - \nu(r_p \vee r_g))$$

Thus

$$|\nu(e_p) - \nu(p)| \leq |\nu(e_p) - \nu(g)| + |\nu(g) - \nu(p)|$$
$$\leq \|e_p - p\|(\nu(e_p) - \nu(r_p \vee r_g)) + \|e_p - p\|(\nu(p) - \nu(e_p + r_p \vee r_g - p))$$
$$\leq 2\|e_p - p\|(1 - 2\|e_p - p\|)^{-1}(\nu(e_p) - \nu(r_p \vee r_g))$$

(b) Now, let $\nu_0 \neq 0$. Without loss of generality we may suppose that $\nu_0(e) = \tau(Ee_+)$, $\forall e \in P$. Here, E is an operator affiliated to the center of $\mathcal{A}$. Denote $|E| = (E^2)^{1/2}$. The restriction of $\overline{\nu}_0(.) \equiv \tau(|E|J.)$ to the projections of any W^*J-factor is a linear indefinite measure. Clearly

$$0 \leq \overline{\nu}_0(e_p) = -\overline{\nu}_0(r_g) = -\overline{\nu}_0(r_p) \leq \overline{\nu}_0(Q^+) = \|\nu_0\|(I)$$

By Corollary 3(ii), $eJ \geq F_e$, $\forall e \in P^+$. So, $\nu_0(e) = \tau(EF_e) \leq \tau(|E|eJ) = \overline{\nu}_0(e)$. Hence $0 \leq \nu(e) \leq \nu_1(e) + \overline{\nu}_0(e)$, $\forall e \in P^+$, and $0 \geq \nu(e) = \nu_1(e) \geq \nu_1(e) + \overline{\nu}_0(e)$, $\forall e \in P^-$. Thus $\nu_1 + \overline{\nu}_0$ is a linear indefinite measure on any W^*J-factor of type I_2. By (a).

$$\begin{aligned}|\nu(e_p) - \nu(p)| &= |\nu_1(e_p) - \nu_1(p)| \\ &\leq |\nu_1(e_p) + \overline{\nu}_0(e_p) - \nu_1(p) - \overline{\nu}_0(p)| + |\overline{\nu}_0(e_p) - \overline{\nu}_0(p)| \\ &\leq 2\|e_p - p\|(1 - 2\|e_p - p\|)^{-1}[\nu_1(e_p) + \overline{\nu}_0(e_p) \\ &\quad - \nu_1(r_p \vee r_g) - \overline{\nu}_0(r_p \vee r_g)] + \|e_p - p\|[\overline{\nu}_0(e_p) - \overline{\nu}_0(r_p)] \\ &\leq 2\|e_p - p\|(1 - 2\|e_p - p\|)^{-1}[\nu(Q^+) - \nu(Q^-) + 5\|\nu\|(I)]\end{aligned}$$

2. Now, consider the general case of $p \in P^+$. There exists a decomposition $p = p_1 + p_2 + p_3$ with $p_i \in P^+$ such that $e_{pi} \preccurlyeq Q^- \wedge (F_{pi} \vee e_{pi})$, $\forall i$, and

$\|e_{pi} - p_i\| \le \|e - p\|$. For $\|e_{pi} - p_i\|$ the estimate was obtained in part 1 of the proof.

Lemma 11. Let $\nu: P \to R$ be an indefinite measure on a W^*J-algebra $\mathcal{A}$. If $\mathcal{A}$ is a W^*K-algebra, then ν is a linear measure on any W^*J-factor $\mathcal{B} \subset \mathcal{A}$ of type I_2. If $\mathcal{A}$ is a W^* P-algebra containing no type $I_{n,m}$ $(n, m \le 2)$ direct summands, then there exists a semiconstant measure ν_0 such that $\nu - \nu_0$ is a linear measure on any W^* J-factor $\mathcal{B} \subset \mathcal{A}$ of type I_2.

Proof. Let $\mathcal{B} \subset \mathcal{A}$ be a W^*J-factor of type I_2. Let $g_+ \in P^+ \cap \mathcal{B}$ and $g_- \in P^- \cap \mathcal{B}$ be such that $g \equiv g_+ + g_-$ is the identity in $\mathcal{B}$. Choose a canonical symmetry $J_1 \in \mathcal{A}$ with the properties $g_+ \le 1/2(I + J_1)$ and $g_- \le 1/2(I - J_1)$. In the Hilbert space H with the inner product $(x, y)_1 \equiv [J_1x, y]$, g_+ and g_- are orthogonal projections and there exists a partial isometry $w \in \mathcal{B}$ with the initial projection g_+ and the final one g_-.

If $\mathcal{A}$ is a W^*K-algebra, then there exists a W^*K-factor $M \subset \mathcal{A}$ of type I_∞ such that $\mathcal{B} \subset M$. By the Theorem,[13] the restriction of ν to $P \cap M$ is a linear measure. Hence the restriction of ν to $P \cap \mathcal{B}$ ($\subset P \cap M$) is a linear measure, too.

Now, let $\mathcal{A}$ be a W^*P-algebra and τ a faithful normal semifinite trace on $\mathcal{A}$. Let $e \in P$, $e \le Q^+$. Denote by V_e the set of all partial isometries $\nu \in V$ with the initial projection e.

(i) Assume that there exist orthogonal projections f, f_1, f_2 such that $e \sim f \sim f_1 \sim f_2$, $f + f_1 + f_2 \le Q^-$. Let v, w_1, and $w_2 \in V_e$ be such that $vv^* = f$ and $w_i^*w_i = f_i$, $i = 1, 2$. The W^*J-algebra $\mathcal{A}(v, w_1, w_2)$ generated by v, w_1, and w_2 is a W^*J-factor of type $I_{1,3}$. (Every $p \in P^+ \cap \mathcal{A}(v, w_1, w_2)$, $p > 0$ is an atom in this factor.) By the Theorem,[13] there exists a unique constant $c(e)$ such that the restriction of the function μ, where $\mu(g) \equiv \nu(g)$ if $g_+ = 0$ and $\mu(g) \equiv \nu(g) - c(e)$ if $g_+ \ne 0$, is a linear measure on $\mathcal{A}(v, w_1, w_2)$. We shall show that μ is a linear measure on $\mathcal{A}(u)$, $\forall u \in V_e$. By the Theorem,[13] we have the following: If a measure is linear on some W^*J-factor of type I_2 in $B(H)$, then it is linear. The measure μ is linear on the W J-factor $\mathcal{A}(w_1)$ of type I_2. Hence μ is linear on the W^*J-factor $\mathcal{A}(v_0, w_1)$ of type I_3 for every $v_0 \in V_e$, $v_0v_0^* = f$. Thus μ is linear on $\mathcal{A}(v_0)$. Let $v_1 \in V_e$ be such that $v_1v_1^* \le Q^- - f$. The measure μ is linear on $\mathcal{A}(v) \subset \mathcal{A}(v, w_1, w_2)$. Hence μ is linear on $\mathcal{A}(v, v_1)$. Therefore, μ is linear on $\mathcal{A}(v_1)$. Let $v_2 \in V_e$ and $F_{v_2} \wedge f = 0 = F_{v_2} \wedge (Q^- - f)$. By the assumption, $f + f_1 + f_2 \le Q^-$. Hence there exists $v_1 \in V_e$ such that $F_{v_1} \perp f$ and $F_{v_1} \perp F_v = 0$. The measure μ is linear on $\mathcal{A}$ (v_1). Hence μ is linear on $\mathcal{A}(v_1, v_2)$. Thus μ is linear on $\mathcal{A}(v_2)$. Next, we similarly establish that μ is linear on $\mathcal{A}(u)$, $\forall u \in V_e$.

(ii) In the general case, there is a decomposition $e = \Sigma_1^n e_i$, $e_i \le Q^+$, such that for every e_i condition (i) is fulfilled. Put $c(e) = \Sigma_1^n c(e_i)$. Obviously

μ with the constant $c(e)$ is a linear measure on $\mathscr{A}(v)$. Hence $c(e)$ does not depend on a decomposition of e.

(iii) In the same way, we can see that $c(\bar{e}) = c(e)$ if $\bar{e} \sim e$, $\bar{e} \leq Q^+$.

(iv) Let us establish that $c(\Sigma e_i) = \Sigma c(e_i)$. Here, the convergence of a sum of an uncountable family of summands means that there only exists a countable set of nonzero terms in the family and the usual series with these summands converges absolutely. Suppose the contrary. Assume that there exists a sequence $\{e_n\}$ of mutually orthogonal projections such that $c(e_n) > 0$. $\forall n$ [or $c(e_n) < 0$. $\forall n$], and $\Sigma c(e_n) = +\infty$ [or $\Sigma c(e_n) = -\infty$]. Let f_n be a J-self-adjoint linear functional on $\mathscr{A}(ve_n)$ such that $\nu(p) = f_n(g) + c(e_n)$, $\forall g \in P^+ \cap \mathscr{A}(ve_n)$, and $\nu(p) = f_n(g)$, $\forall g \in P^- \cap \mathscr{A}(ve_e)$. Put $\psi_n \equiv ve_n v^*$. The operators

$$q(3/2\psi_n, \pm v^* \psi_n) \equiv 1/2(3\psi_n \pm \sqrt{3}v^*\psi_n \mp \sqrt{3}\psi_n v - v^*\psi_n v)$$

are projections in P^-. Hence

$$\begin{aligned}\nu(q(3/2\psi_i, \pm v^*\psi_i)) &= 1/2(3f_n(\psi_n) \pm \sqrt{3}f_n(v^*\psi_n - \psi_n v)) - f_n(e_n)) \\ &= 1/2(3\nu(\psi_n) \pm 2\sqrt{3}\Re f_n(v^*\psi_n) - (\nu(e_n) - c(e_n)))\end{aligned}$$

Let $X \equiv \{n: (\nu(e_n) - c(e_n))\Re f_n(v^*\psi_n) \leq 0\}$. The projections in

$$\{q(3/2\psi_n, v^*\psi_n)\}_{n\in X} \cup \{q(3/2\psi_n, -v^*\psi_n)\}_{n\in N\setminus X}$$

are mutually orthogonal by the construction. Hence there exists a projection

$$q \equiv \sum_{n\in X} q(3/2\psi_n, v^*\psi_n) + \sum_{n\in N\setminus X} q(3/2\psi_n, -v^*\psi_n) \in P^-$$

This implies

$$M \equiv 1/2\left(\sum_{n\in N} \left|\nu(q(3/2\psi_n, v^*\psi_n)\right| + \sum_{n\in N\setminus X} \left|\nu(q(3/2\psi_n, -v^*\psi_n)\right|\right) < +\infty$$

From this, it follows that

$$\begin{aligned}0 &\leq 1/2 \sum_{n\in N} \left|\nu(e_e) - c(e_e)\right| \\ &\leq 1/2(\sum_{n\in X} \left|2\sqrt{3}\Re f_n(v^*\psi_n) - (\nu(e_n) - c(e_n))\right| \\ &\quad + \sum_{n\in N\setminus X} -2\sqrt{3}\Re f_n(v^*\psi_n) - (\nu(e_n) - c(e_n))\big|) \\ &\leq M + 3/2 \sum_n \left|f_n(\psi_n)\right| = M + 3/2 \sum_n \left|\nu(\nu_n)\right| < +\infty\end{aligned}$$

In addition. $0 \leq \Sigma\, \nu(e_n) < +\infty$. Hence $+\infty = \Sigma\, c(e_n) < +\infty$. We have a contradiction. Thus $c(e) = \Sigma_i\, c(e_i)$ for any decomposition $e = \Sigma\, e_i$.

The function $c(e)$ is a completely additive measure. There exists an integrable in the trace τ self-adjoint operator T affiliated to the W^*-algebra $Q^+\mathcal{A}Q^+$ such that $c(e) = \tau(Te)$, $\forall e$.[9,7] We proved that $c(e) = c(f)$ if $e \sim f$. Hence the operator T is affiliated to the center of $Q^+\mathcal{A}Q^+$. By the Corollary of ref. 5, p. 18), there is an operator Z affiliated to the center of $\mathcal{A}$ such that $T = Q^+ZQ^+$. Denote by ν_0 the semiconstant measure $\tau(Zg_+)$, $g \in P$.

(v) It remains to prove that $\nu - \nu_0$ is linear on every W^*J-factor $\mathcal{B} \subset \mathcal{A}$ of type I_2. Let f be the identity in $\mathcal{B}$. Fix some decomposition $f = f_+ + f_-$, $f_+ \in P^+$, $f_- \in P^-$. Choose a canonical symmetry $J_1 \in \mathcal{A}$ such that $f_+ \leq Q_1^+ \equiv 1/2\ (I + J_1)$. Then f_+ and f_- are orthogonal projections with respect to the metric $(x, y)_1 \equiv [J_1, x, y]$.

(a) Assume first that there exist e, $r \in P$, $e \leq Q_1^+ \wedge Q^+$, $r \leq 1/2(I - J_1) \wedge Q^-$ such that $f(e + r) = 0$, $e \sim f_+$, and $e \sim r$. Let v and w be partial isometries with respect to $(., .)_i$ with $v^*v = e$, $vv^* = r$, and $w^*w = f_+$, $ww^* = e$. Then $\mathcal{A}(\mathcal{B}, v, w)$ is a W^*J-factor of type I_4. By (iii), $c(f_+) = c(e)$. Therefore, $\nu - \nu_0$ is a linear measure on $\mathcal{A}(\mathcal{B}, v, w)$. Hence $\nu - \nu_0$ is linear on $\mathcal{B}$.

(b) Note that by the assumption on $\mathcal{A}$ [see (i)], (a) is fulfilled if f_+ is an Abelian projection. For $\mathcal{B}$, there exists a finite set of W^*J-factors $\mathcal{B}_i$ of type I_2 such that for $\mathcal{B}_i$, $\forall i$, condition (a) is fulfilled; $ab = 0$, $\forall a \in \mathcal{B}_i$, $\forall b \in \mathcal{B}_i$, $i \neq j$; $\mathcal{B} \subset \oplus_i\mathcal{B}_i$. The measure $\nu - \nu_0$ is linear on $\mathcal{B}_i$, $\forall i$. Hence the measure $\nu - \nu_0$ is linear on $\mathcal{B}$.

Corollary 12. Every indefinite measure on a W^*J-algebra $\mathcal{A}$ without type $I_{n,m}$ (n, $m \leq 2$) direct summands is continuous on P^+ and P^- in the norm operator topology.

Proof. The W^*J-algebra $\mathcal{A}$ decomposes uniquely into two direct summands. $\mathcal{A}_1$, $\mathcal{A}_2$, where $Q^+\mathcal{A}_1Q^+$ has no type I_2 direct summand and $Q^+\mathcal{A}_2Q^+$ is of type I_2. (Note that $Q^-\mathcal{A}_2Q^-$ is a direct sum of type I_n, $3 \leq n \leq +\infty$, summands.) The restriction of ν to $P^+ \cap Q^+\mathcal{A}_1Q^+$ is a completely additive measure. By the Theorem,[9,7] ν is continuous on $P^+ \cap Q^+\mathcal{A}_1Q^+$ in the norm topology. Also, ν has a similar property on the projections in $Q^+\mathcal{A}_2Q^+$.

For every $e \in P^+$ the operator $J_1 \equiv e_+(I - e)_+ - (I - e)_-$ is a canonical symmetry. With respect to the inner product $(x, z)_1 \equiv [J_1x, z]$, H is a Hilbert space, e is an orthogonal projection, and $e \in Q_1^+\mathcal{A}Q_1^+$ [here, $Q_1^+ \equiv 1/2(I - J_1)$]. Let $\{p_n\} \subset P^+$ and $\|p_n - e\|_1 \to 0$. Then $\|e_{p_n} - e\|_1 \to 0$ and $\|p_n - e_{p_n}\|_1 \to 0$. By Lemmas 10 and 11, $\lim_{n\to\infty} \nu(p) = \nu(e)$. Hence ν is continuous on P^+ in the norm operator topology. Analogously, ν is continuous on P^-.

If ν is an indefinite measure, then the function $\nu^*(p) \equiv \nu(p^*)$. $\forall p \in P$, is an indefinite measure also.

Remark 13. The indefinite measures ν and ν^* have the same semiconstant measure ν_0 satisfying Lemma 11.

For every measure ν and its Hermitian component the semiconstant measure ν_0 satisfying Lemma 11 is the same. A semiconstant measure cannot be linear. Therefore, the following proposition singles out the linear Hermitian component of a measure ν. It completely describes indefinite Hermitian measures.

Proposition 14. Let ν be a Hermitian indefinite measure on a W^*J-algebra $\mathcal{A}$ without type $I_{n,m}$ $(n, m \leq 2)$ direct summands. Define a measure ν_0 by setting $\nu_0 \equiv 0$ if $\mathcal{A}$ is a W^*K-algebra and taking ν_0 as in Lemma 11 if $\mathcal{A}$ is a W^*P-algebra. Denote by ν_+ and ν_- the self-adjoint ultraweakly linear functionals with the supports not exceeding Q^+ and Q^- such that $\nu(p) = \nu_+(p) + \nu_0(p)$. $\forall p \leq Q^+$ and $\nu(p) = \nu_-(p) + \nu_0(p)$, $\forall p \leq Q^-$. Then $\nu(p) = \nu_+(p) + \nu_-(p)\ \nu_0(p)$. $\forall p \in P$.

Proof. Let $p = p(\alpha e, v) \in P^+$. $\mathcal{A}(e, v)$ is a W^*J-factor of type I_2. Hence $\nu - \nu_0$ is a linear Hermitian measure on $\mathcal{A}(e, v)$. By Section 3 there exists a self-adjoint and J-self-adjoint linear functional $f \in \mathcal{A}_*(e, v)$ such that $\nu(p) - \nu_0(p) = f(p)$, $\forall p \in P \cap \mathcal{A}(e, v)$. By Proposition 6,

$$\begin{aligned}\nu(p) = f(p) + \nu_0(p) &= f(Q^+pQ^+) + f(Q^-pQ^-) + \nu_0(p)\\ &= \nu_+(p) + \nu_-(p) + \nu(p).\end{aligned}$$

By Remark 4, Corollary 12, and Proposition 9, the equality (8) is true for every $p \in P^+ \cup P^-$ and hence for every $p \in P$.

Corollary 15. The skew-Hermitian component of every indefinite measure is linear on any W^*J-subfactor of type I_2 and is continuous on $P^+ \cup P^-$ in the norm topology.

5. SOME PROPERTIES OF A SKEW-HERMITIAN MEASURE

Lemma 16. Let μ be the skew-Hermitian component of an indefinite measure ν. Assume that ν is linear on any W^*J-subfactor $\mathcal{A}(u)$, $u \in V$, of type I_2. Then

$$\left|\mu(p(\alpha e, u))\right| \leq 2(\alpha(\alpha - 1))^{1/2}(-\nu(u^*u)\nu(uu^*))^{1/2}$$

Proof. We may identify $\mathcal{A}(u)$ with the algebra of all (2, 2) matrices in a J-space H, dim $H = 2$. Let $\nu(.) = tr(T.)$. By Section 3, we may assume that $TJ \geq 0$. Let

$$T = \begin{pmatrix} a & b \\ -\overline{b} & c \end{pmatrix}$$

in an orthonormal base $(e.\phi)$. $e \in H^+ \cup S$. $\phi \in H^- \cap S$. Since $JT \geq 0$. we get $|b|^2 \leq -ac = -\nu(u^*u)\nu\,(uu^*)\ [\leq -\nu(Q^+)\nu(Q^-)]$. There exists $\theta \in R$ such that

$$p(\alpha e, u) = \begin{pmatrix} \alpha & -e^{i\theta}(\alpha(\alpha - 1))^{1/2} \\ e^{-i\theta}(\alpha(\alpha - 1))^{1/2} & \alpha - 1 \end{pmatrix}$$

in the base $(e.\ \phi)$. Hence

$$\begin{aligned} |\mu(p(\alpha e, u))| &= |tr(1/2(T - T^*)p(\alpha e, u))| \\ &= |2\Re e^{-i\theta}b|(\alpha(\alpha - 1))^{1/2} \\ &\leq 2|b|(\alpha(\alpha - 1))^{1/2} \leq 2(\alpha(\alpha - 1))^{1/2}\,(-\nu(u^*\,u)\nu uu^*))^{1/2} \end{aligned}$$

Lemma 17. Let μ be a skew-Hermitian measure. For every projection $p \in P^-$ there exists a projection $f \in P^+$ such that $\mu(p) = -\mu(f) = \mu(f^*)$.

Proof. Let $p \in P^-$ and $q \equiv p - p \wedge Q^-$. Then $f \equiv F_{Q^+q} + F_{Q^-q} - q \in P^+$ and $\mu(f) + \mu(p) = \mu(f) + \mu(q) = 0$. Hence $\mu(p) = \mu(q) = -\mu(f) = \mu(f^*)$.

Lemma 18. Let μ be the skew-Hermitian component of an indefinite measure ν, and $p = p(\alpha e, v)$. Then there is a constant c such that $|\mu(q)| < c$, $\forall q \leq p$.

Proof. By the linearity of μ (see Corollary 15) on $\mathscr{A}(v)$, there is a skew-adjoint and J-adjoint linear functional $\overline{\mu}$ on $\mathscr{A}(v)$ such that $\mu(p(\beta e, v)) = \overline{\mu}(p(\beta e, v))$, $\forall p(\beta e, v)) \in \mathscr{A}(v)$. Note that

$$\begin{aligned} \mu(p(\alpha e, v)) &= (\alpha^2 - \alpha)^{1/2}\,\overline{\mu}(v - v^*) \\ &= (\alpha^2 - \alpha)^{1/2}\,(\beta^2 - \beta)^{-1/2}\mu(p(\beta e, v)) \end{aligned}$$

By Corollary 15, for a given $t > 0$ there exists $\delta > 0$ such that $|\mu(g)| = |\mu(Q^+) - \mu(g)| < t$ if $\|Q^+ - g\| < \delta$. There exists $\beta > 0$ such that $\|Q^+ - p(\beta Q^+, u)\| < \delta$ for every partial isometry u with the initial projection Q^+ and the final one $F_u \leq Q^-$. Let $g \leq p = p(\alpha e, v)$. Then $g = p(\alpha e_g, ve_g)$.

(a) Let $\mathscr{A}$ be a W^*P-algebra. Then there exists a partial isometry w with the initial projection $Q^+ - e_q$ and the final one $ww^* \leq Q^- - ve_gv^*$. Since $\mu(r) = -\mu(r^*)$. $\forall r \in P$, and $p^*(\alpha e, v) = p(\alpha e, -v)$, if follows that

$$\text{either} \quad \mu(g)\mu(p(\alpha(Q^+ - e_g), w) \geq 0 \quad \text{or} \quad \mu(g)\mu(p(\alpha(Q^+ - e_g), -w)) \geq 0$$

Let, for example, the first be true. Then

$$\begin{aligned} |\mu(g)| &\leq |\mu(p(\alpha Q^+, ve_g + w))| \\ &= (\alpha^2 - \alpha)^{1/2}(\beta^2 - \beta)^{-1/2}|\mu(p(\beta Q^+, ve_g + w))| \\ &\leq (\alpha^2 - \alpha)^{1/2}(\beta^2 - \beta)^{-1/2}t \end{aligned}$$

(b) Let $\mathcal{A}$ be a W^*K-algebra. By Lemma 11, ν is linear on any W^*J-factor $\mathcal{B} \subset \mathcal{A}$ of type I_2. By Lemma 16,

$$\left|\mu(g)\right| = \left|\mu(p(\alpha e_g, ve_g))\right| \leq 2(\alpha(\alpha - 1))^{1/2}(-\nu(Q^+)\nu(Q^-))^{1/2}$$

The proof is complete.

Put $p = p(\alpha e, v) \in P^+$. By Corollary 3,

$$\|p\| = 2\alpha - 1,$$

$$F_p = (2\alpha - I)^{-1}(\alpha e + (\alpha^2 - \alpha)^{1/2}(v + v^*) + (\alpha - 1)vv^*)$$

and $p = (2\alpha - 1)F_pJ$. For every orthogonal projection $f \in \Pi$, $f \leq F_p$, the operator $g \equiv (2\alpha - 1)fJ$ is a projection in P^+ and $g \leq p$. Also, $F_g = f$ and $p(\alpha e_g, ve_g) = g$.

Next, let μ be the skew-Hermitian component of an indefinite measure. Let $\gamma_v^\alpha(f) \equiv (2\alpha - 1)^{-1}\ \mu((2\alpha - 1)fJ)$, $\forall f \in \Pi$, $f \leq F_p$. By Corollary 3(iv) and the definition of a measure on P, we have $\gamma_v^\alpha(f) = \Sigma\ \gamma_v^\alpha(f_i)$ for any decomposition $f = \Sigma f_i$. By Lemma 18, $\left|\gamma_v^\alpha(e)\right| \leq c$ for some constant c. By the Theorem,[11] it follows from the additivity and the boundedness of γ_v^α that there exists a Hermitian linear functional $\overline{\gamma}_v^\alpha \in \mathcal{A}_*$ such that $\overline{\gamma}_v^\alpha(f) = \gamma_v^\alpha(f)$, $\forall f \in \Pi$, $f \leq F_p$ and the support of $\overline{\gamma}_v^\alpha$ does not exceed F_p. Thus

$$\overline{\gamma}_v^\alpha(gJ) = \overline{\gamma}_v^\alpha((2\alpha - 1)F_g) = (2\alpha - 1)\overline{\gamma}_v^\alpha(F_g)$$

$$= (2\alpha - 1)\gamma_v^\alpha(F_g) = \mu((2\alpha - 1)F_gJ) = \mu(g), \qquad \forall g \in P^+, \quad g \leq p$$

Let $w_\alpha \equiv (2\alpha - 1)^{-1/2}(\alpha^{1/2}e + (\alpha - 1)^{1/2}v)$. It can be easily verified that $w_\alpha^* w_\alpha = e$ and $w_\alpha w_\alpha^* = F_{p(\alpha e, v)}$, i.e., w_α is a partial isometry with the initial projection e and the final one $F_{p(\alpha e, v)}$. Let $\overline{\mu}$ be a skew-adjoint and J-adjoint linear functional such that $\overline{\mu}(p) = \mu(p)$, $\forall p_\alpha \equiv p(\alpha e, v) \in \mathcal{A}(v)$. Then

$$\overline{\gamma}_v^\alpha(w_\alpha e w_\alpha^*) = \gamma_v^\alpha(F_{p_\alpha}) = (2\alpha - 1)^{-1}\mu(p_\alpha)$$

$$= (2\alpha - 1)^{-1}(\alpha^2 - \alpha)^{1/2}\overline{\mu}(v - v^*)$$

$$= (2\alpha - 1)^{-1}(\alpha^2 - \alpha)^{1/2}(\beta^2 - \beta)^{-1/2}\mu(p_\beta)$$

$$= (2\alpha - 1)^{-1}(\alpha^2 - \alpha)^{1/2}(\beta^2 - \beta)^{-1/2}(2\beta - 1)\overline{\gamma}_v^\beta(F_{p_\beta})$$

$$= (2\alpha - 1)^{-1}(\alpha^2 - \alpha)^{1/2}(\beta^2 - \beta)^{-1/2}(2\beta - 1)\overline{\gamma}_v^\beta\ (w_\beta e w_\beta^*)$$

Thus

$$(2\alpha - 1)(\alpha^2 - \alpha)^{-1/2}\ \overline{\gamma}_v^\alpha(w_\alpha b w_\alpha^*)$$

$$= (2\beta - 1)(\beta^2 - \beta)^{-1/2}\overline{\gamma}_v^\beta(w_\beta b w_\beta^*), \qquad \forall b \in \mathcal{A}(v)$$

Hence the linear functional $(2\alpha - 1)(\alpha^2 - \alpha)^{-1/2}\overline{\gamma}_v^\alpha(w_\alpha, w_\alpha^*)$ does not depend on α. Let $\overline{\gamma}_v(.) = 2^{-1}(2\alpha - 1)(\alpha^2 - \alpha)^{-1/2}\overline{\gamma}_v^{\alpha}(w_\alpha, w_\alpha^*)$. By the definition, the support of $\overline{\gamma}_v$, does not exceed Q^+.

Lemma 19. Let $\overline{\mu}_v(.) \equiv \overline{\gamma}_v(v^*. - .v)$. The function $\overline{\mu}_v$ is a linear skew-Hermitian measure on P, and $\overline{\mu}_v(q) = \mu(q)$ holds for all projections $q = p(x, vF_x)$ and $q = p(x, -vF_x)$ [$= p^*(x, vF_x)$].

Proof. By (2) and (3),

$$Q^+(v^*g - gv)^*Q^+ = Q^+(v^*g - gv)\,Q^+$$
$$Q^+(v^*g^* - g^*v)Q^+ = -Q^+(v^*g - gv)^*Q^+$$

$\forall g \in P$. Thus $\overline{\mu}_v(g) \in R$ and $\overline{\mu}_v(g) = -\overline{\mu}_v(g^*)$, $\forall g \in P$. Hence $\overline{\mu}_v(.)$ is a linear skew-Hermitian measure on P.

It is easy to see that

$$w_\alpha F_q = (2\alpha - 1)^{-1/2}\alpha^{1/2}F_q, \qquad F_q\, vw_\alpha^* = (2\alpha - 1)^{-1/2}(1 - \alpha)^{1/2}F_q$$

for every orthogonal projection $F_q \le F_{p(\alpha e, v)}$. Hence for $q \le p(\alpha e, v)$ we have

$$\begin{aligned} w_\alpha v^* q w_\alpha^* &= w_\alpha v^* F_q q w_\alpha^* = (2\alpha - 1)^{-1/2}(\alpha - 1)^{1/2}F_q qJJw_\alpha^* \\ &= (2\alpha - 1)^{-1/2}(\alpha - 1)^{1/2}(2\alpha - 1)F_q Jw_\alpha^* = (2\alpha - 1)^{1/2}(\alpha - 1)^{1/2}F_q w_\alpha^* \\ &= (\alpha - 1)^{1/2}\alpha^{1/2}F_q \end{aligned}$$

and

$$-w_\alpha q v w_\alpha^* = -w_\alpha F_q qJJvw_\alpha^* = (\alpha - 1)^{1/2}\alpha^{1/2}F_q$$

Thus

$$\begin{aligned} \overline{\mu}_v(q) &= (2\alpha - 1)2^{-1}(\alpha\,(\alpha - 1))^{-1/2}\overline{\gamma}_v^{\alpha}(w_\alpha(v^*q - qv)w_\alpha^*) \\ &= (2\alpha - 1)2^{-2}(\alpha(\alpha - 1))^{-1/2}\overline{\gamma}_v^{\alpha}(2(\alpha - 1)^{1/2}\alpha^{1/2}F_q) \\ &= (2\alpha - 1)\overline{\gamma}_v^{\alpha}(F_q) = \mu(q) \end{aligned}$$

$\forall q \in P^+$, $q \le p(\alpha e, v)$. It follows that

$$\begin{aligned} \overline{\mu}_v(p(\beta e_q, ve_q)) &= (\beta^2 - \beta)^{1/2}\,\overline{\mu}_v(ve_q - e_q\, v^*) \\ &= (\beta^2 - \beta)^{1/2}(\alpha^2 - \alpha)^{-1/2}\,\overline{\mu}_v(p(\alpha e_q, ve_q)) \\ &= (\beta^2 - \beta)^{1/2}(\alpha^2 - \alpha)^{-1/2}\,\mu(p(\alpha e_q, ve_q)) = \mu(p(\beta e_q, ve_q)) \end{aligned}$$

$\forall p(\beta e_q, ve_q)$ and $\forall q = p(\beta e_q, ve_q) \le p(\alpha e, v)$. Finally, by Remark 4 and Corollary 12, we have

$$\mu(p(x, vF_x)) = \overline{\mu}_v(p(x, vF_x)) = -\overline{\mu}_v(p(x, -vF_x)) = -\mu(p(x, -vF_x))$$

The proof is complete.

It follows from Lemma 19 and Corollary 15 that $M_\alpha^\mu < +\infty$, $\forall\alpha > 1$. Note that $(\alpha^2 - \alpha)^{-1/2}M_\alpha^\mu = (\beta^2 - \beta)^{-1/2}M_\beta^\mu$, $\forall\alpha, \beta$.

Let $\overline{\gamma}_v = |\overline{\gamma}_v|(e^+ - e^-)$ be the polar decomposition for $\overline{\gamma}_v$. Here, $e^+, e^- \in \Pi$ and $e^+ + e^- = e \le Q^+$. We have $Q^+(v^*p - pv)Q^+ = (x^2 - x)^{1/2} + (x^2 - x)^{1/2}$, $\forall p = p(x, v) \in P^+$. Hence

$$0 \le Q^+(v^*p - pv)Q^+ = 2(x^2 - x)^{1/2}$$
$$\le 2(\alpha^2 - \alpha)^{1/2}\, Q^+ \qquad \text{if} \qquad \|x\| \le \alpha$$

By the latter inequalities, we have

$$\begin{aligned}|\mu(p(x, v)| &= |\,\overline{\gamma}_v(2e^+(x^2 - x)^{1/2}e^+) - \overline{\gamma}_v(2e^-(x^2 - x)^{1/2}e^-)| \\ &\le \overline{\gamma}_v(2(\alpha^2 - \alpha)^{1/2}e^+) - \overline{\gamma}_v(2(\alpha^2 - \alpha)^{1/2}e^-) \\ &= \mu(p(\alpha e^+, ve^+)) - \mu(p(\alpha e^-, ve^-)) \\ &= \mu(p(\alpha e^+, ve^+)) + \mu(p(\alpha e^-, -ve^-)) \\ &= \mu(p(\alpha e, v(e^+ - e^-))) \le M_\alpha^\mu\end{aligned}$$

Let $\|\mu\|(p) \equiv \sup\{\Sigma|\mu(p_i)|:p = \Sigma p_i\}$. Put $f^+ \equiv w_\alpha e^+ w_\alpha^*$ and $f^- \equiv w_\alpha e^- w_\alpha^*$. Then

$$\overline{\gamma}_v^\alpha = (\overline{\gamma}_v^\alpha(f^+, f^+) + \overline{\gamma}_v^\alpha(f^-, f^-)$$

is the polar decomposition of $\overline{\gamma}_v^\alpha$. For every $q \equiv p(\alpha e_q, ve_q)$, $g \equiv p(\alpha e_g, ve_g)$, and $q, g \le p(\alpha e, v)$, we have $qg = gq = 0 \Leftrightarrow e_q e_g = 0 \Leftrightarrow F_q F_g = 0$. If follows that

$$\begin{aligned}\sum_i |\mu(p_i)| &= (2\alpha - 1)\sum_i |\overline{\gamma}_v^\alpha\,(F_{p_i})| \\ &\le (2\alpha - 1)\sum_i |\overline{\gamma}_v^\alpha|\,(F_{p_i}) = (2\alpha - 1)|\overline{\gamma}_v^\alpha|(\sum_i F_{p_i}) = (2\alpha - 1)|\overline{\gamma}_v^\alpha|(F_p) \\ &\le (2\alpha - 1)\|\overline{\gamma}_v^\alpha\| = (2\alpha - 1)(\overline{\gamma}_v^\alpha(f^+) - \overline{\gamma}_v^\alpha(f^-)) \\ &= \mu(p(\alpha e, v(e^+ - e^-)))\end{aligned}$$

for any representation $p(\alpha e, v) = \Sigma p_i$. Let $s \equiv e^+ - e^-$. Observe that the functional $\overline{\mu}_{vs}$ is positive.

Corollary 20. (*i*) $\|\mu\|(p(\alpha e, v)) = (2\alpha - 1)\|\overline{\gamma}_v^\alpha\|$. In particular, if $\|\mu\|(p(\alpha e, v)) = 0$, then $\overline{\gamma}_v^\alpha \equiv 0$, and if $M_\alpha^\mu = 0$, then $\mu \equiv 0$.

(ii) $M_\alpha^\mu = \sup\{\mu(p): p = p(\alpha Q^+, v)\}$.

Proof. Part (i) is obvious.

(ii) By Corollary 3 (i), $\|p(x, w)\| = \|p(\|x\|F_x, w)\|$. Obviously, $|\mu(p(\beta e, w))| \le \mu(p(\alpha e, w))|$ if $\beta \in [1, \alpha)$. Hence by Corollary 15,

$$\left|\mu(p(x, w))\right| \leq \sup\{\mu\ (p(\|x\|F_x, u))\}$$

the supremum being taken over all $u \in V$, with $u^*u = F_x$.

(a) Let $\mathcal{A}$ be a W^* P-algebra. By the assumption, $Q^+ \preccurlyeq Q^-$. Hence for every $w \in V$, $w^*w \neq Q^+$, there exists $u \in V$ such that $u^*u = Q^+ - w^*w$ and $F_u \perp F_w$. The measure μ is skew-Hermitian. Thus either

$$\begin{aligned}\mu(p(\alpha Q^+, w + u) &= \mu(p(\alpha w\ w, w)) + \mu(p(\alpha(Q^+ - w^*w), u))\\ &\geq \mu(p(\alpha w^*w, w))\end{aligned}$$

or

$$\begin{aligned}\mu(p(\alpha Q^+, w - u)) &= \mu(p(\alpha w^*w, w)) + \mu(p^*(\alpha(Q^+ - w^*w), - u))\\ &\geq \mu(p(\alpha w^*w, w))\end{aligned}$$

It follows that

$$\mu(p(\alpha e, w)) \leq \sup\{\mu(p)\colon p = p(\alpha Q^+, v) \in P^+\} \quad (\leq M^{\mu}_{\alpha})$$

for every projection $p(\alpha e, w) \in P$. Thus Part (ii) is fulfilled.

(b) Let $\mathcal{A}$ be a W^*K-algebra. The inequalities proved above are also satisfied here. The following situation is also possible. For a given $\epsilon > 0$ there exists $p(\alpha e, v)$ such that $v^*v = e \neq Q^+$, $F_v = Q^-$, and $M^{\mu}_{\alpha} - \mu(p(\alpha e, v)) < \epsilon$. Let $e_n \downarrow 0$ be a sequence of orthogonal projections such that $e_n \leq e$, $e_n \sim Q^-$. By the assumption, we have $Q^+ \preccurlyeq Q^-$. Consequently, there is a sequence of partial isometries $\{v_n\} \subset V$ with the initial projections $Q^+ - (e - e_n)$ and the final ones ve_nv^* $[= Q^- - v(e - e_n)v^*]$, $\forall n$. Then $p_n \equiv p(\alpha Q^+, v(e - e_n) + v_n(Q^+ - (e - e_n)) \in P^+$. By Corollary 15 and Lemma 16,

$$\begin{aligned}&\left|\mu(p_n) - \mu(p(\alpha e, v))\right|\\ &\quad = \left|\mu(p(\alpha(Q^+ -(e - e_n)), v_n(Q^+ - (e - e_n))) - \mu(p(\alpha e_n, ve_n))\right|\\ &\quad \leq 2(\alpha(\alpha - 1))^{1/2}[-\nu(Q^+) - (e - e_n))^{1/2}\ \nu(vev^*)^{1/2}\\ &\qquad - \nu(e_n)^{1/2}\nu(ve_nv^*)^{1/2}] \to 0\end{aligned}$$

if $n \to +\infty$. Hence there exists n such that $M^{\mu}_{\alpha} - \mu(p_n) < \epsilon$. Thus part (ii) follows.

Lemma 21. Let $v \in V$ be a partial isometry such that $M^{\mu}_{\alpha} - \mu(p(\alpha Q^+, v)) < \epsilon$, $\epsilon \in (0.1)$. Assume that $Q^- - F_v > 0$. Then for any $p(\alpha e, w)$, where $F_w \leq Q^- - F_v$, the following inequality holds: $\left|\mu(p(\alpha e,w))\right| \leq \max\{\epsilon, 2\epsilon^{1/2}(M^{\mu}_{\alpha})^{1/2}\}$.

Proof. Without loss of generality we may assume that

$$(0 \leq) \quad \|\mu\|(p(\alpha e, w)) = \mu(p(\alpha e, w)) \quad \text{and}$$
$$\|\mu\|(p(\alpha Q^{+}, v)) = \mu(p(\alpha Q^{+}, v))$$

and that $M_{\alpha}^{\mu} > 0$ [see Corollary 20(i)].

Consider the W^*J-factor $\mathcal{B} = \mathcal{A}(w, ve)$ of type I_3. By the linearity of the restriction of μ to $P \cap \mathcal{B}$ and by Proposition 7, there exists a nonnegative linear functional $\phi \in \mathcal{B}_*$ with the support $f \leq e$, $f \neq 0$, such that $\mu(p) = \phi(u^*p - pu)$, $\forall\, p \in P \cap \mathcal{B}$. Here, $u \in \mathcal{B}$ is a partial isometry with the initial projection e and the final one $F_u \leq F_w + F_{ve}$.

If $F_u \perp F_w$, then $e(u^*p(\alpha e, w) - p(\alpha e, w)u)e = 0$. Hence $\mu(p(\alpha e, w)) = 0$. If $F_u = F_w$, then $\mu(p(\alpha e, ve)) = 0$. Hence $0 \leq \mu\,(p(\alpha e, w)) < \epsilon$. Now, let $F_u \neq F_w$ and $F_u \neq F_{ve}$. By Lemma 2.3,[7]

$$F_{ve} = \beta F_u + (\beta - \beta^2)^{1/2}\,(v_0 + v_0^*) + (1 - \beta)v_0 v_0^*$$

where $0 < \beta < 1$ and v_0 is the partial isometry from the polar decomposition $(I - F_u)F_{ve}F_u = v_0\left|(I - F_u)F_{ve}\, F_u\right|$. It can be easily verified that $\beta^{1/2}F_u + (1 - \beta)^{1/2}v_0^* \in \mathcal{B}$ is a partial isometry with the initial projection F_{ve} and the final one F_u, and $(1 - \beta)^{1/2}F_u - \beta^{1/2}v_0^* \in \mathcal{B}$ is a partial isometry with the initial projection F_w and the final one F_u. Therefore,

$$u^*ve = u^*(F_uF_{ve})ve = \beta^{1/2}u^*(\beta^{1/2}F_u + (1 - \beta)^{1/2}v_0^*)ve = \beta^{1/2}e^{i\theta}e$$

and $u^*w = u^*F_uF_ww = (1 - \beta)^{1/2}e^{it}e$ for some complex number $e^{i\theta}$ and e^{it}. Thus

$$\begin{aligned}\mu(p(\alpha e, ve)) &= \phi((\alpha^2 - \alpha)^{1/2}\,(u^*ve + ev^*u)) \\ &= 2(\alpha^2 - \alpha)^{1/2}\Re e^{i\theta}\beta^{1/2}\phi(e)\end{aligned}$$

and $\mu(p(\alpha e, w)) = 2(\alpha^2 - \alpha)^{1/2}\Re e^{it}(1 - \beta)^{1/2}\phi(e)$. By the assumption of Lemma 21, we have

$$2(\alpha^2 - \alpha)^{1/2}(1 - \Re^{i\theta}\beta^{1/2})\phi(e) = \mu(p(\alpha e, u) - \mu(p(\alpha e, ve) < \epsilon$$

Hence

$$1 - \beta^{1/2} \leq 1 - \beta^{1/2}\Re e^{i\theta} < \epsilon(2(\alpha^2 - \alpha)^{1/2}\phi(e))^{-1}$$

Finally,

$$\begin{aligned}(0 \leq) \quad \mu(p(\alpha e, w)) &\leq 2(\alpha^2 - \alpha)^{1/2}(1 - \beta)^{1/2}\phi(e) \\ &= 2(\alpha^2 - \alpha)^{1/2}(1 + \beta^{1/2})^{1/2}(1 - \beta^{1/2})^{1/2}\phi(e) \\ &\leq 2(\alpha^2 - \alpha)^{1/2}(1 + \beta^{1/2})^{1/2}(\epsilon(2(\alpha^2 - \alpha)^{1/2}\phi(e))^{-1})^{1/2}\phi(e) \\ &\leq 2\epsilon^{1/2}(2(\alpha^2 - \alpha)^{1/2}\phi(e))^{1/2} \leq 2\epsilon^{1/2}(M_{\alpha}^{\mu})^{1/2}\end{aligned}$$

It remains to make use of the arguments in Section 2.

Lemma 22. If for every $\delta > 0$ there exists a partial isometry $v \in V$ such that $\left|\mu(p) - \overline{\mu}_v(p)\right| < \delta\|p\|$ for every simple projection $p \in P$, then the measure μ is linear.

Proof. Let, for a given $\delta > 0$, $v \in V$ be a partial isometry from the assumption of Lemma 22. Let $\gamma_v(e) \equiv \overline{\gamma}_v(v^*e + ev)$, $\forall e \in \Pi$. Note that $\gamma_v(e) = 0$ if $e = e \wedge Q^+ + e \wedge Q^-$. Also,

$$\left|\gamma_v(e)\right| \leq 2\|\overline{\gamma}_v\| = (\alpha^2 - \alpha)^{-1/2}(2\alpha - 1)\|\overline{\gamma}_v^{\alpha}\| \leq (\alpha^2 - \alpha)^{-1/2} M_\alpha^\mu$$

Let $M^\mu \equiv (\alpha^2 - \alpha)^{-1/2} M_\alpha^\mu$. First we define a measure γ on Π. Let $e \in \Pi$. Put $\gamma(e) \equiv \overline{\gamma}_u(u^*e + eu)$, where u is the partial isometry from the polar decomposition $Q^-eQ^+ = u\left|Q^-eQ^+\right|$ if $Q^-eQ^+ \neq 0$ and $\gamma(e) \equiv 0$ if $e = e \wedge Q^+ + e \wedge Q^-$. Hence $\left|\gamma(e)\right| \leq M^\mu$, $\forall e \in \Pi$.

Now we will estimate $\left|\gamma(e) - \gamma_v(e)\right|$. One may suppose that $e \wedge Q^+ = e \wedge Q^- = 0$. Let $x \equiv Q^+eQ^+ = \int \lambda de_\lambda$ be the spectral resolution of Q^+eQ^+. By Lemma 2.3,[7] $e = x + u(x^2 - x)^{1/2} + (x^2 - x)^{1/2}u^* + u(\mathrm{I} - x)u^*$. For every number n denote by $x_{k/n}$ the operator $[(2k - 1)/(2n)](e_{k/n} - e_{k-1/n})$. Let

$$e_{k/n} = x_{k/n} + u(x_{k/n}^2 - x_{k/n})^{1/2} + (x_{k/n}^2 - x_{k/n})^{1/2}u^* + u(e_{k/n} - e_{k-1/n} - x_{k/n})u^*$$

and $e_n \equiv \Sigma_1^n e_{k/n}$. By the construction, $\|e - e_n\| < 12n^{-1/2}$. Hence

$$\max\{\left|\gamma(e) - \gamma(e_n)\right|, \left|\gamma_v(e - \gamma_v(e_n)\right|\} \leq M^\mu\|e - e_n\| \leq M^\mu 12n^{-1/2}$$

By the construction, $1/2$ is a regular point for $x_n \equiv \Sigma_1^n x_{k/n}$. Thus

$$g_{k/n} \equiv 2\left(\frac{2k - 1}{2n} - 1\right)^{-1} e_{k/n} J \in P, \quad \forall k \quad \text{and} \qquad g_{k/n}g_{i/n} = 0 \quad (k \neq i)$$

It is clear that

$$\|g_{k/n}\| = \left(\frac{2k - 1}{n} - 1\right)^{-1}$$

By the linearity of $\overline{\gamma}_u$,

$$\begin{aligned}
\gamma(e_n) &= \overline{\gamma}_u(u^*e_n + e_n u) \\
&= \sum_{k=1}^{n} \|g_{k/n}\|^{-1}\overline{\gamma}_u(u^*g_{k/n} J + g_{k/n} Ju) \\
&= \sum_{k=1}^{n} \|g_{k/n}\|^{-1}\, \overline{\gamma}_u(u^*g_{k/n} - g_{k/n}u) \\
&= \sum_{k=1}^{n} \|g_{k/n}\|^{-1}\mu(g_{k/n})
\end{aligned}$$

By the assumption of Lemma 21, $\left|\mu(g_{k/n}) - \overline{\mu}_v(g_{k/n})\right| \leq \delta\|g_{k/n}\|$. Hence

$$\left|\gamma(e_n) - \gamma_v(e_n)\right| = \left|\sum_{k=1}^{n} \|g_{k/n}\|^{-1} \mu(g_{k/n}) - \sum_{k=1}^{n} \|g_{k/n}\|^{-1} \overline{\mu}_v(g_{k/n})\right| = n\delta$$

Finally,

$$\begin{aligned}\left|\gamma(e) - \gamma_v(e)\right| &\leq \left|\gamma(e) - \gamma(e_n)\right| + \left|\gamma(e_n) - \gamma_v(e_n)\right| + \left|\gamma(e_n) - \gamma_v(e)\right| \\ &\leq M^{\mu}12n^{-1/2} + n\delta + M^{\mu}12n^{-1/2}\end{aligned}$$

From the latter we obtain that $\gamma(.)$ is continuous in 0 in the strong operator topology.

Now, let $e = f + r$, where $f, r \in \Pi$. Then $\gamma_v(e) = \gamma_v(f) + \gamma_v(r)$ and

$$\begin{aligned}\left|\gamma(e) - \gamma(f) - \gamma(r)\right| &\leq \left|\gamma(e) - \gamma_v(e)\right| + \left|\gamma(f) - \gamma_v(f)\right| + \left|\gamma(r) - \gamma_v(r)\right| \\ &\leq 2M^{\mu}12n^{-1/2} + n\delta\end{aligned}$$

It follows that $\gamma(e) = \gamma(f) + \gamma(r)$.

Thus γ is bounded additive and continuous in 0 in the strong operator topology function on the logic Π. Hence γ is a measure on Π. Again, by the Theorem,[11] there exists a self-adjoint functional $\overline{\gamma} \in \mathcal{A}_*$ such that $\gamma(e) = \overline{\gamma}(e)$, $\forall e \in \Pi$.

Let now $p \in P$. Then 1/2 is a regular point for $F_pQ^+F_p$. By Section 2, $p = \int_{-\infty}^{1+} (2\lambda - 1)^{-1}\, d(f_\lambda F_p)J$, where f_λ is the projection from the spectral resolution $F_pQ^+ F_p = \int \lambda df_\lambda$. Let u be the partial isometry in the polar decomposition $Q^-pQ^+ = u\left|Q^-pQ^+\right|$. By Lemma 19,

$$\begin{aligned}\mu(p) &= \overline{\gamma}_u(u^*p - pu) = \overline{\gamma}_u(u^*pJ + pJu) \\ &= \overline{\gamma}_u(u^* \int_{-\infty}^{1+} (2\lambda - 1)^{-1}\, d(f_\lambda F_p) + \int_{-\infty}^{1+} (2\lambda - 1)^{-1}\, d(f_\lambda F_p)u) \\ &= \int_{-\infty}^{1+} (2\lambda - 1)^{-1}\, d(\overline{\gamma}_u(u^*(f_\lambda F_p) + (f_\lambda F_p)u) \\ &= \int_{-\infty}^{1+} (2\lambda - 1)^{-1}\, d\gamma(f_\lambda) \\ &= \overline{\gamma}\left(\int_{-\infty}^{1+} (2\lambda - 1)^{-1}\, d(f_\lambda F_p)\right) = \overline{\gamma}(pJ)\end{aligned}$$

6. THE PROOF OF THE THEOREM

By Corollary 20(ii), for every $\epsilon > 0$ there exists $p(\alpha Q^+, v)$ such that $M^{\mu}_{\alpha} - \mu(p(\alpha Q^+, v)) < \epsilon$. Now, we will show that the assumption of Lemma 22 is satisfied.

(i) Let $\{e_j\}$ be a finite orthogonal set of projections $e_j \leq Q^+$. Let $\{\beta_j\}$ and $\{\alpha_j\}$, where $\alpha_j > 1$ are sets of real numbers. Denote by p the projection $\Sigma\, p(\alpha_j e_j, e^{i\beta_j} v e_j)$ and by $\mathcal{B}$ the W^*J-algebra $\oplus_j \mathcal{A}(e_j, ve_j)$. Obviously $\mathcal{B}$ is a direct sum of W^*J-factors of type I_2. Hence by Corollary 15, the restriction of μ to $P \cap \mathcal{B}$ extends to a linear functional on $\mathcal{B}$. Obviously $p \in \mathcal{B}$ and $M^{\mu}_{\alpha}(\mathcal{B}) - \mu(p(\alpha\, \Sigma\, e_j, v(\Sigma\, e_j))) < \epsilon$. By Proposition 8, there exists a function $\delta(\epsilon)$ satisfying $\delta(\epsilon)_{\epsilon\to 0} \to 0$ such that

$$\left|\mu(p) - \mu_v(p)\right| \leq \|p\|\delta(\epsilon) \tag{9}$$

(ii) Let $s \in V$ be a partial isometry with the initial subspace eH and the final one veH. The restriction of sv^* to veH is a unitary operator. By making use of the spectral resolution of sv^* on veH, (i), and the norm continuity of μ and μ_v, we obtain (9) for $q \equiv p(\beta e, s)$.

(iii) Let $s \in V$ be a partial isometry with the initial subspace eH. Assume that $sH \subseteq vH$ and $(sH) \perp (veH)$. The algebra $\mathcal{B} \equiv \mathcal{A}(vf, ve, v^*s)$, where $f = v^*ss^*v$ is a W^*J-factor of type I_4, and $p(\beta e, s) \in P \cap \mathcal{B}$. By Proposition 8 and the Theorem,[13] (9) is satisfied for $p(\beta e, s)$.

(iv) Let $e, f \in \Pi$, $e \vee f \leq Q^+$, and $0 = e^{\perp} \wedge f = e \wedge f^{\perp} = e \wedge f$. By Lemma 2.3,[7]

$$e = x + w(x - x^2)^{1/2} + (x - x^2)^{1/2}\, w^* + w(I - x)w^*$$

where $x = fef$ and w is the partial isometry in the polar decomposition $f^{\perp}ef = w\left|f^{\perp}ef\right|$. Let $x = \int_0^{1+} \lambda\, df(\lambda)$ be the spectral resolution for x and

$$\Delta f_k \equiv f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right), \qquad x_{k/n} \equiv \frac{k + 1/2}{n}\, \Delta f_k, \qquad x_n \equiv \sum_0^{n-1} x_{k/n}$$

Also, let

$$e_{k/n} \equiv x_{k/n} + w(x_{k/n} - x_{k/n}^2)^{1/2} + (x_{k/n} - x_{k/n}^2)^{1/2}\, w^* + w(\Delta f_k - x_{k/n})w^*$$

and $e_n \equiv \Sigma_0^{n-1}\, e_{k/n}$. Obviously, $e_{k/n}, e_n \in \Pi$. Denote by $\mathcal{B}$ the W^*J-algebra $\oplus_0^{n-1} \mathcal{A}\,(w\Delta f_k, v\Delta f_k, vw\Delta f_k)$. Clearly $\mathcal{B}$ is a direct sum of W^*J-factors of type I_4. Again, by Corollary 15, the restriction of μ to $P \cap \mathcal{B}$ extends to a linear functional on $\mathcal{B}$. In addition, we have $q \equiv p(\alpha(f \vee e), v(f \vee e)) \in P \cap \mathcal{B}$ and $M^{\mu}_{\alpha}(\mathcal{B}) - \mu(q) \leq \epsilon$. Let $u_n \equiv x_n^{1/2} + w(f - x_n)^{1/2}$ and $u \equiv x^{1/2} + w(f - x)^{1/2}$. Note that u_n (resp. u) is a partial isometry with the initial projection f and the final projection e_n (e, respectively), and $p(\beta f, vu_n) \in \mathcal{B}$. By the Theorem[13] and Proposition 8, (9) holds for $p(\beta f, vu_n)$. Observe that

$$\|p(\beta f, vu) - p(\beta f, vu_n)\|_{n-\infty} \to 0$$

Hence by Corollary 15, the inequality (9) holds for $p(\beta f, vu)$.

Let $e \wedge f \neq 0$ and $0 = e^{\perp} \wedge f = e \wedge f^{\perp}$. Then $\oplus_0^{n-1}\mathcal{A}(w\Delta f_k, v\Delta f_k, vw\Delta f_k) \oplus \mathcal{A}(v(e \wedge f))$ is a sum of W^*J-algebras of type I_4 and a W^*J-factor of type I_2. Hence (9) holds for $p(\beta f, v(u + e \wedge f))$.

Let $e^{\perp} \wedge f \neq 0$ and $e \wedge f^{\perp} \neq 0$. Suppose that there exists a partial isometry u_0 with the initial and final projections $e^{\perp} \wedge f$ and $e \wedge f^{\perp}$. Thus by (iii), the inequality (9) holds for $p(\beta f, v(u_0 + u + e \wedge f))$.

(v) Let $\mathcal{A}$ be a W^*K-algebra.

(a) Let first $s \in V$ be such that $ss^* = vv^*$ and $s^*s \equiv e < Q^+$. As in the proof of Corollary 20, there exists a sequence $e_n \downarrow 0$ of orthogonal projections $e_n \leq e$ with $Q^+ - e + e_n \sim se_ns^*$. Let $\{s_n\} \subset V$ be a sequence with the initial projections $Q^+ - e + e_n$ and the final ones se_ns^*, $\forall n$. Denote $p_n = p(\beta Q^+, s(e - e_n) + s_n (Q^+ - (e - e_n))$. By the complete additivity of the measures μ and μ_v, we have

$$\mu(p(\beta e, s)) = \lim_{n\to\infty} \mu(p(\beta e(e - e_n), s(e - e_n)))$$

$$\mu_v(p(\beta e, s)) = \lim_{n\to\infty} \mu_v(p(\beta e(e - e_n), s(e - e_n)))$$

By Lemma 11 and Lemma 16,

$$\lim_{n\to\infty}[\mu(p(\beta e(e - e_n), s(e - e_n))) - \mu(p_n)] = 0$$

$$\lim_{n\to\infty}[\mu_v(p(\beta e(e - e_n), s(e - e_n))) - \mu_v(p_n)] = 0.$$

Hence by (ii).

$$\left|\mu(p(\beta e, s) - \mu_v(p(\beta e, s))\right| = \lim \left|\mu(p_n) - \mu_v(p_n)\right| \leq \|p(\beta e, s)\|\delta(\epsilon)$$

(b) Let $s \in V$ be such that $s^*s = Q^+$ and $ss^* < vv^*$. Similarly, (9) holds true for $p(\beta Q^+, s)$.

Lemma 23. For every $w \in V$ the restriction of μ to $P \cap E\mathcal{A}E$, where $E \equiv w^* w + ww^*$, is a linear measure.

Proof. Without loss of generality we may assume that $w^*w = Q^+$. Let, for a given $\epsilon > 0$, a projection $p(\alpha Q^+, y)$, $yy = ww^*$, be such that $M_\alpha^\mu(E\mathcal{A}E) - \mu(p(\alpha Q^+, y)) < \epsilon$.

Consider a projection $p(\beta e, s) \in P$, $seH \subseteq yQ^+H$. Assume that $seH = yeH$ (or $seH \perp yeH$). By analogy with (ii) and (iii), we obtain $\left|\mu(p) - \mu_g(p)\right| \leq \|p\|\delta(\epsilon)$ for $p = p(\beta e, s)$.

Now, consider the general case of $p(\beta e, s)$. Denote by f the projection $y^*(F_{ye} \vee F_{se} - F_{ye})y$ and by r the projection $y^*(F_{ye} \vee F_{se})y$. Note that $F_{ye} \sim F_{se}$.

(a) Let $\mathcal{A}$ first be a W^*P-algebra. Then there exists a partial isometry u_0 with the initial projection $f \wedge r^{\perp}$ and the final one $f^{\perp} \wedge r$. Let u be the partial isometry from (iv). Note that u has the initial projection $f - f \wedge r^{\perp}$ and the final one $r - r \wedge f^{\perp}$. By the definition, $y(e + f)H = yeH \vee seH = seH \oplus y(u + u_0)fH$. The operator $(s + y(u + u_0))$ is a partial isometry with the initial and the final projections $e + f$ and $F_{ye} \vee F_{se}$, respectively. Define $q \equiv p(\beta e, s) + p(\beta f, y(u + u_0))$. By analogy with (ii). $|\mu(q) - \mu_y(q)| \le \|q\|\delta(\epsilon) = (2\beta - 1)\delta(\epsilon)$. By analogy with (iii) and (iv), we have

$$\left|\mu(p(\beta f, y(u + u_0))) - \mu_y(p(\beta f, y(u + u_0)))\right| \le 2(2\beta - 1)\delta(\epsilon) = 2\|q\|\delta(\epsilon)$$

Thus

$$\begin{aligned}
&\left|\mu(p(\beta e, s)) - \mu_y(p(\beta e, s))\right| \\
&\quad = \left|\mu(q) - \mu(p(\beta f, y(u + u_0))\right. \\
&\qquad \left. + \mu_v(p(\beta f, y(u + u_0)) - \mu_y(q)\right| \\
&\quad \le \left|\mu(q) - \mu_y(q)\right| + \left|\mu(p(\beta f, y(u + u_0)) - \mu_y(p(\beta f, y(u + u_0))\right| \\
&\quad \le 3(2\beta - 1)\delta(\epsilon) = 3\|p(\beta e, s)\|\delta(\epsilon)
\end{aligned}$$

(b) Let $\mathcal{A}$ be a W^*K-algebra. The cases $F_{ye} \le F_{se}$ and $F_{se} \le F_{ye}$ (i.e., $r = 0$ and $f = 0$) were examined in (ii) and (v). Now, let $r \neq 0$, $f \neq 0$, and u be the partial isometry in (a). We again examine two cases.

(1) Suppose that there is a partial isometry d with the initial and final projections $f \wedge r^{\perp}$ and $r' \le r \wedge f^{\perp}$ (i.e., $f \wedge r^{\perp} \preccurlyeq r \wedge f^{\perp}$). Hence the partial isometry $y(d + u)$ has the initial projection f and the final one $\le F_{ye} \vee F_{se} - F_{se}$. By analogy with (iii) and (iv), the inequality (10) holds for $p(\beta f, y(d + u))$. The operator $y(d + u) + s$ belongs to V and has the initial projection $f + e$ and the final one $\le f_{ye} \vee F_{se}$. By analogy with (v), step (a), (10) holds for $p(\beta(f + e), y(d + u) + s)$. By analogy with (a), (10) holds for $p(\beta e, s)$.

(2) Suppose that there is a partial isometry d_0 with the initial, and final projection, $r \wedge f^{\perp}$ and $f' \le f \wedge r^{\perp}$ (i.e., $r \wedge f^{\perp} \preccurlyeq f \wedge r^{\perp}$). By analogy with (v), step (b), (9) holds for $p(\beta(d_0 d_0^* + f - f \vee r^{\perp}), y(d_0 + u) + s)$. Hence for $p(\beta e, s)$, (10) holds.

Now, consider the general case of $f \wedge r^{\perp}$ and $r \wedge f^{\perp}$. By ref. 5, Theorem 1, p. 218, there exists a central projection $G \in \mathcal{A}$ such that $G(f \wedge r^{\perp}) \preccurlyeq G(r \wedge f^{\perp})$ and $(I - G)$ $(r \wedge f^{\perp} \le (I - G)(f \wedge r^{\perp})$. Hence by (b1) and (b2), we have

$$\begin{aligned}
&\left|\mu(p(\beta e, s)) - \mu_y(p(\beta e, s))\right| \\
&\quad \le \left|\mu(p(\beta eG, sG)) - \mu_y(p(\beta eG, sG))\right| \\
&\qquad + \left|\mu(p(\beta eG^{\perp}, sG^{\perp})) - \mu_y(p(\beta eG^{\perp}, sG^{\perp}))\right| \le 6\|p(\beta e, s)\|\delta(\epsilon)
\end{aligned}$$

Thus for a given $\delta > 0$ there exists $\epsilon > 0$ such that

$$M^{\mu}_{\alpha}(E\mathscr{A}E) - \mu(p(\alpha Q^{+}, y)) < \epsilon \qquad \text{implies } \left|\mu(p) - \mu_{y}(p)\right| \leq \|p\|\delta$$

for every simple projection $p(\beta e, s)$. In view of Lemma 22, the proof of Lemma 23 is complete.

(vi) Now we finish the proof of the Theorem.

Let first $seH \subseteq vQ^{+}H$. By Lemma 23 and Propositions 8 and 7, for a given $\delta > 0$ there exists $\epsilon > 0$ such that $0 \leq M^{\mu}_{\alpha} - \mu(p(\alpha Q^{+}, v)) < \epsilon$ implies $\left|\mu(p(\beta e, s)) - \mu_{v}(p(\beta e, s))\right| \leq \|p(\beta e, s)\|\delta(\epsilon)$.

Now, suppose that seH is not a subset of $vQ^{+}H$. Every $g = p(\beta e, s)$ is representable as a sum $p(\beta e, s) = p(\beta e_1, se_1) + p(\beta e_2, se_2)$, where $e_1 \equiv v^*(f_v \wedge F_{se})v$ and $e_2 \equiv e - e_1$. By Lemma 23 and Corollary 15, the following holds:

$$\left|\mu(p(\beta e_1, se_1)) - \mu_{v}(p(\beta e_1, se_1))\right| \leq 3(2\beta - 1)\delta(\epsilon) = 3\|p(\beta e, s)\|\delta(\epsilon)$$

There is a decomposition $e_2 = e_2^1 + e_2^2 + e_2^3$, where $e_2^i, i = 1, 2, 3,$ are orthogonal projections, $e_2^1 \sim e_2^2$, and e_2^3 is Abelian. By the construction, there exist an orthogonal projection p_i with $e_2^i \leq p_i \leq Q^{+}$ and $w_i \in V$ with the initial projection p_i and the final one $w_i w_i^* \geq (se_2^i s^*) \vee (ve_2^i v^*)$ $(i = 1, 2)$. By Lemma 23, the restriction of μ to $P \cap F\mathscr{A}F$, where $F \equiv p_i + w_i p_i w_i^*$ and hence to $P \cap G\mathscr{A}G$, where $G \equiv e_2^i + (ve_2^i v^*) \vee (se_2^i s^*)$, is a linear measure. By Proposition 8, we have

$$\left|\mu(g_i) - \mu_{v}(g_i\right| \leq \|p(\beta e, s)\|\delta(\epsilon), \qquad g_i \equiv p(\beta e_2^i, se_2^i)\ (i = 1, 2)$$

Let $g_3 \equiv p(\beta e_2^3, se_2^3)$. Now we will show that the inequality (10) is true for g_3. The W^*J-algebra $\mathscr{A}(se_2^3, ve_2^3)$ is of type I_3 (if $e_2^3 \neq 0$). Put $f \equiv ve_2^3 v^*$ and $r \equiv se_2^3 s^*$. Obviously, f and r are abelian projections. Let Δf_k $(k = 0, n - 1, u,$ and u_n be as in step (iv). By (ii), we may assume that f and r are the initial and the final projections for u. The operator $w \equiv u^* s v^*$ is a unitary one in the Abelian algebra $f\mathscr{A}f$. Hence for a given $n \in N$ there exists a finite set of mutually orthogonal projections $f_j \leq f$ $(j = 1, \ldots, m)$ and real numbers β_j such that for $w_n \equiv \Sigma_1^m e^{i\beta_j} f_j$ the inequality $\|w - w_n\| \leq 1/n$ holds. Put $f_{kj} \equiv f_k \Delta f_j$ and $s_n \equiv u_n w_n v$. By the construction, the W^*J-algebra $\mathscr{B} \equiv \oplus_{k,j} \mathscr{A}(v^* f_{kj}, u_n f_{kj})$ is a direct sum of W^*J-factors of type I_3, and $p(\alpha e_2^3, ve_2^3) \in \mathscr{B}$. We have $\|s - s_n\| \leq 3/n$. By the Theorem,[13] the restriction of μ to $P \cap \mathscr{B}$ is a linear measure. Hence by the norm continuity of μ and μ_v (see Corollary 12), we have

$$\begin{aligned}
&\left|\mu(p(\beta e_2^3, se_2^3)) - \mu_{v}(p(\beta e_2^3, se_2^3))\right| \\
&\quad = \lim_{n-\infty} \left|\mu(p(\beta e_2^3, s_n e_2^3)) - \mu_{v}(p(\beta e_2^3, s_n e_2^3))\right| \\
&\quad \leq (2\beta - 1)\delta(\epsilon) = \|p(\beta e, s)\|\, \delta(\epsilon)
\end{aligned}$$

Thus

$$\left|\mu(g) - \mu_v(g)\right| \leq \left|\mu(p(\beta e_1, se_1)) - \mu_v(p(\beta e_1, se_1))\right| + \sum_1^3 \left|\mu(g_i) - \mu_v(g_i)\right| \leq 7\,\|g\|\,\delta(\epsilon)$$

Now, let

$$q \equiv q((\beta - 1)ww^*, w^*) = \beta w^*w + (\beta^2 - \beta)^{1/2}(w - w^*) - (\beta - 1)ww^*$$

be a simple negative projection [see (3)]. Then $p \equiv p(\beta ww^*, w^*)$ is a simple positive projection. Also, $p \perp q$ and $p + q = ww^* + w^*w \in \Pi$. We have $\mu(p) + \mu(q) = \mu(ww^* + w^*w) = 0 = \mu_v(p) + \mu_v(q)$. Hence

$$\left|\mu(q) - \mu_v(q)\right| = \left|\mu(p) - \mu_v(p)\right| \leq 7\|p\|\,\delta(\epsilon) = 7\|q\|\,\delta(\epsilon)$$

Thus the assumption of Lemma 22 is satisfied. By Lemma 22, μ is a linear measure. This concludes the proof of the Theorem.

Remark 24. Similar to Π and L, for an arbitrary W^*J-algebra of type I_2 there exists an indefinite measure on P such that the Theorem fails to be true. We believe that the theorem is valid for the W^*J-algebras of types $I_{1,2}$ and $I_{2,2}$ (for the W^*J-factors of these types the Theorem is known to be true[14]).

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